

DATKO-TYPE THEOREMS CONCERNING ASYMPTOTIC BEHAVIOUR OF EXPONENTIAL TYPE IN MEAN

Pham Viet Hai

Communicated by P.A. Cojuhari

Abstract. In this paper, we study the concept of exponential (in)stability in mean for stochastic skew-evolution semiflows, in which the exponential (in)stability in the classical sense is replaced by an average with respect to a probability measure. Our paper consists of three major results. The first is to obtain Datko-type characterizations for the exponential stability in mean of stochastic skew-evolution semiflows. Next, we acquire Datko-type characterizations for the exponential instability in mean by extending the stability techniques. The last is to extend Lyapunov-type equations to the case of exponential (in)stability in mean.

Keywords: exponential stability, exponential instability, stochastic skew-evolution semiflows, Datko's theorem, Banach function spaces.

Mathematics Subject Classification: 93E15, 34D05, 46E30.

1. INTRODUCTION

The solutions of a nonautonomous linear Cauchy problem on a Banach space $\mathbb{X}$ can be described, under suitable assumptions, by a two-parameter family $\{U(q, t)\}_{q \geq t \geq 0}$ included in the Banach space $B(\mathbb{X})$ of bounded linear operators on $\mathbb{X}$, for which the following conditions hold for every $p_1 \geq p_2 \geq p_3 \geq 0$: (i) $U(p_1, p_1) = I$ (the identity operator); (ii) $U(p_1, p_3) = U(p_1, p_2)U(p_2, p_3)$; (iii) for each $x \in \mathbb{X}$, the mapping $(p_1, p_2) \mapsto U(p_1, p_2)x$ is continuous. This family is known as an *evolution family*. In particular, if $U(p_1 - p_2, 0) = U(p_1, p_2)$ for every $p_1 \geq p_2 \geq 0$, then the one-parameter family $\{T(\lambda)\}_{\lambda \geq 0}$ defined by $T(\lambda) = U(\lambda, 0)$ is called a *C_0 -semigroup*. We say that the family $\{U(q, t)\}_{q \geq t \geq 0}$ is *exponentially bounded* if there are constants M, ε with the property

$$\|U(\alpha, \beta)\| \leq M e^{(\alpha - \beta)\varepsilon}, \quad \forall \alpha \geq \beta \geq 0,$$

where the symbol $\|\cdot\|$ indicates the norm on $\mathbb{X}$ and on $B(\mathbb{X})$. Other notations will be explained in Subsection 2.1. If $\{U(q, t)\}_{q \geq t \geq 0}$ is exponentially bounded with some $\varepsilon < 0$, then we say that it is *exponentially stable*.

In [3], Datko showed that a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on a Hilbert space $\mathcal{H}$ is exponentially stable if and only if there exists a bounded linear operator $Q \geq 0$ such that

$$\langle Gz, Qz \rangle + \langle Qz, Gz \rangle = -\|z\|^2, \quad \forall z \in \text{dom}(A), \quad (1.1)$$

where G denotes the generator of $\{T(t)\}_{t \geq 0}$. It was proved in [12] that the equation (1.1) has the following equivalent form

$$Q - T^*(d)QT(d) = \int_0^d T^*(\tau)T(\tau) d\tau, \quad \forall d \geq 0. \quad (1.2)$$

The Lyapunov functionals in Hilbert spaces are crucial to Datko's proof techniques, and we note that they arise from the following result also obtained by Datko in [3].

Theorem 1.1 ([3]). *A C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on a Hilbert space $\mathcal{H}$ is exponentially stable if and only if for every $y \in \mathcal{H}$ the orbit $\lambda \mapsto \|T(\lambda)y\|$ belongs to the Lebesgue function space $L^2(\mathbb{R}_{\geq 0})$; meaning that*

$$\int_0^\infty \|T(\lambda)y\|^2 d\lambda < \infty, \quad \forall y \in \mathcal{H}.$$

Datko's results were seminal, inspiring numerous studies on the asymptotic theory of dynamical systems, including Pazy's extension [11] to the more general $L^d(\mathbb{R}_{\geq 0})$ spaces ($d \geq 1$), Datko's extension [4] to evolution families. The results presented in [3, 4] specifically addressed the scenario involving $L^2(\mathbb{R}_{\geq 0})$. However, later developments showed that the conclusion extends to the case where $L^2(\mathbb{R}_{\geq 0})$ is replaced by $L^d(\mathbb{R}_{\geq 0})$. This generalization led to the following refinement of Datko's theorem.

Theorem 1.2 ([8]). *Let $\{U(q, t)\}_{q \geq t \geq 0}$ be an exponentially bounded evolution family on a Banach space $\mathbb{X}$. Then the family $\{U(q, t)\}_{q \geq t \geq 0}$ is exponentially stable if and only if there exists $d \in \mathbb{R}_{\geq 1}$ such that for every $y \in \mathbb{X}$ the mapping $\lambda \mapsto \|U(\lambda + \eta, \eta)y\|$ belongs to the Lebesgue function space $L^d(\mathbb{R}_{\geq 0})$ in the uniform sense; meaning that*

$$\sup_{\eta \geq 0} \int_0^\infty \|U(\lambda + \eta, \eta)y\|^d d\lambda < \infty, \quad \forall y \in \mathbb{X}.$$

Zabczyk's discrete-time work [17], though perhaps less known, is significant for showing that if $\mathbf{Z} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a convex, increasing, continuous function vanishing at 0, and $B \in L(\mathbb{X})$ such that for each $y \in \mathbb{X}$ there is a positive constant $\alpha(y) > 0$ with the property that

$$\sum_{j \geq 0} \mathbf{Z}(\alpha(y)\|B^j y\|) < \infty,$$

then the spectral radius of B is less than 1. Due to convenience for testing assumptions and for computation, the research of discrete-time versions has attracted many experts. Of these, we can recall a particular form obtained in [9].

Theorem 1.3 ([9]). *Let $\{U(q, t)\}_{q \geq t \geq 0}$ be an exponentially bounded evolution family on a Banach space $\mathbb{X}$. Then $\{U(q, t)\}_{q \geq t \geq 0}$ is exponentially stable if and only if there exists $d \in \mathbb{R}_{\geq 1}$ such that for each $y \in \mathbb{X}$ the mapping $k \mapsto \|U(k + \lambda, \lambda)y\|$ belongs to the Lebesgue sequence space $\ell^d(\mathbb{Z}_{\geq 0})$ in the uniform sense; meaning that*

$$\sup_{\lambda \geq 0} \sum_{\eta \in \mathbb{Z}_{\geq 0}} \|U(\eta + \lambda, \lambda)y\|^d < \infty, \quad \forall y \in \mathbb{X}.$$

The function $\mathbf{Z}$ inspired the paper [13] to give a generalization, namely the function $\mathbf{R} : \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfies the properties: (i) $\mathbf{R}(\cdot, s)$ is nondecreasing for all fixed $s \geq 0$; (ii) $\mathbf{R}(t, \cdot)$ is continuous and nondecreasing for all fixed $t > 0$; (iii) $\mathbf{R}(t, 0) = 0$ for every $t > 0$, and $\mathbf{R}(t, s) > 0$ for every $t > 0, s > 0$. We make two comments on the function $\mathbf{R}$. Firstly, it is not necessarily convex, and so it is more general than the function $\mathbf{Z}$. Secondly, it is a generalization of the norms in Lebesgue spaces or Orlicz spaces.

A far-reaching generalization for C_0 -semigroups was done in [15] by Neerven. In details, he demonstrated that a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space is exponentially stable if and only if there exists a Banach function space F with $\lim_{t \rightarrow \infty} |\mathcal{X}_{[0, t]}|_F = \infty$, such that its orbits belong to F . It can be seen that Theorem 1.1 follows from this by taking $F = L^2(\mathbb{R}_{\geq 0})$. Neerven's result was gradually improved and extended: from semigroups to skew-product semiflows ([8]), from Datko-type theorem to Barbashin-type theorem ([5, 6, 16]). Obtained in [9] is a remarkable extension from stability to instability.

Theorem 1.4 ([9]). *Let $\{U(q, t)\}_{q \geq t \geq 0}$ be an exponentially bounded evolution family on a Banach space $\mathbb{X}$. Then $\{U(q, t)\}_{q \geq t \geq 0}$ is exponentially unstable if and only if it is injective and there is $d \in \mathbb{R}_{\geq 1}$ with the property*

$$\sup_{\lambda \geq 0} \int_0^\infty \|U(\alpha + \lambda, \lambda)y\|^{-d} d\alpha \leq Q \|y\|^{-d}, \quad \forall y \in \mathbb{X} \setminus \{0\}.$$

The present paper studies a similar problem for stochastic skew-evolution semiflows with the expected conclusion that the stochastic framework is more delicate to characterize. Our paper consists of three major results. The first one is to obtain Datko-type characterizations for the exponential *stability* in mean of stochastic skew-evolution semiflows. More precisely, we describe the exponential stability in mean, in terms of Banach spaces of functions or sequences. It should be noted that the internal structure of Banach sequence spaces used here is more general than spaces studied in [8, 9]. Next, we acquire Datko-type characterizations for the exponential *instability* in mean by extending the stability techniques. The last is to extend Lyapunov-type equations to the case of exponential (in)stability in mean.

2. PRELIMINARIES

Before coming to the main results, we review some preliminaries on stochastic skew-evolution semiflows and Banach spaces of functions or sequences.

2.1. NOTATIONS

The symbols $\mathbb{C}, \mathbb{R}, \mathbb{Z}$ refer to the sets of complex, real, integer numbers, respectively. Denote $T = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$. For $\lambda \in \mathbb{R}$, denote by $\lfloor \lambda \rfloor$ the greatest integer less than or equal to λ . For $D \subseteq \mathbb{R}$, χ_D is the characteristic function of D , and $D_{\geq \delta} = \{x \in D : x \geq \delta\}$. Let F be the set of all above unbounded, nondecreasing $\vartheta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that $\vartheta(t) = 0 \iff t = 0$. Let $\mathcal{F}$ be the set of all nonincreasing functions $\Lambda : \mathbb{R}_{> 0} \rightarrow \mathbb{R}_{> 0}$ such that $\lim_{t \rightarrow 0} \Lambda(t) = \infty$. Let $\mathcal{S}(\delta)$ be the set of nondecreasing sequences $\{t_n\}$ with $\sup(t_{n+1} - t_n) \leq \delta$.

2.2. SKEW-EVOLUTION SEMIFLOWS

Let $(\mathcal{M}, \mathcal{B}, \mathbf{P})$ be a probability space.

Definition 2.1 ([14]). A pair (ζ, Φ) is called a *stochastic skew-evolution semiflow*, if $\zeta : T \times \mathcal{M} \rightarrow \mathcal{M}$ is a measurable random field (known as a *stochastic evolution semiflow*) satisfying

1. $\zeta(\alpha, \beta, f) = \zeta(\alpha, r, \zeta(r, \beta, f))$, $\forall \alpha \geq r \geq \beta \geq 0$, $\forall f \in \mathcal{M}$,
2. $\zeta(\alpha, \alpha, f) = v$, $\forall \alpha \geq 0$, $\forall f \in \mathcal{M}$,

and the map $\Phi : T \times \mathcal{M} \rightarrow B(\mathbb{X})$ (called as a *stochastic evolution cocycle*) satisfies

1. $\Phi(\alpha, \beta, f) = \Phi(\alpha, r, \zeta(r, \beta, f))\Phi(r, \beta, f)$, $\forall \alpha \geq r \geq \beta \geq 0$, $\forall f \in \mathcal{M}$,
2. $\Phi(\alpha, \alpha, f) = I$, $\forall \alpha \geq 0$, $\forall f \in \mathcal{M}$.

It turns out that stochastic evolution cocycles arise from abstract Cauchy problems.

Example 2.2. Let $Z : \mathcal{M} \rightarrow B(\mathbb{X})$ be a continuous map and $\zeta : T \times \mathcal{M} \rightarrow \mathcal{M}$ be a stochastic evolution semiflow. Further, let $x_z(\cdot)$ be the unique solution of the abstract Cauchy problem

$$x'(t) = Z(\zeta(t, s, v))x(t), \quad x(s) = z.$$

Then the map $\Phi(\cdot, s, v)z = x_z(\cdot)$ is a stochastic evolution cocycle.

What makes a stochastic evolution cocycle interesting is the fact that it can be regarded as a generalization of an evolution family. More interestingly, stochastic evolution cocycles are generalizations of stochastic cocycles (see [5]). Finally, stochastic semiflows give rise from stochastic differential equations (see [1, 2]).

We recall some definitions of stability in the classical sense.

Definition 2.3. A stochastic skew-evolution semiflow (ζ, Φ) is called:

1. *exponentially bounded* if there are constants $M, \varepsilon > 0$ with the property

$$\|\Phi(p, q, f)z\| \leq M e^{\varepsilon(p-q)} \|z\|, \quad \forall (p, q) \in T, z \in \mathbb{X}, f \in \mathcal{M}, \quad (2.1)$$

2. *exponentially stable* if it satisfies (2.1) with $\varepsilon < 0$,
3. *exponentially unstable* if there are constants $M, \varepsilon > 0$ with the property

$$\|\Phi(p, q, f)z\| \geq M e^{\varepsilon(p-q)} \|z\|, \quad \forall (p, q) \in T, z \in \mathbb{X}, f \in \mathcal{M}, \quad (2.2)$$

4. *injective* if for every $(p, q) \in T$, $f \in \mathcal{M}$, $z \in \mathbb{X}$ with $z \neq 0$ we have

$$\|\Phi(p, q, f)z\| > 0.$$

Equipping the probability measure $\mathbf{P}$ on $\mathcal{M}$ can offer the concepts of stability in the stochastic sense. For $\varkappa \geq 1$, $\mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ denotes the Banach space of all Bochner measurable functions $g : \mathcal{M} \rightarrow \mathbb{X}$ with

$$\mathbb{E}_\varkappa[g] := \left(\int_{\mathcal{M}} \|g(\omega)\|^\varkappa d\mathbf{P}(\omega) \right)^{1/\varkappa} < \infty.$$

We identify the functions in $\mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ which are $\mathbf{P}$ -almost everywhere.

We make precise the definitions of stability in the stochastic sense.

Definition 2.4. Let $\varkappa \geq 1$. A stochastic skew-evolution semiflow (ζ, Φ) is called:

1. *exponentially bounded in $\varkappa$ -mean* if there exist positive constants M, ε such that

$$\mathbb{E}_\varkappa[\Phi(t, s, \cdot)g(\cdot)] \leq M e^{\varepsilon(t-s)} \mathbb{E}_\varkappa[g], \quad \forall (t, s) \in T, g \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P}), \quad (2.3)$$

2. *exponentially stable in $\varkappa$ -mean* if it satisfies (2.3) with $\varepsilon < 0$, that is

$$\mathbb{E}_\varkappa[\Phi(t, s, \cdot)g(\cdot)] \leq M e^{-\varepsilon(t-s)} \mathbb{E}_\varkappa[g], \quad \forall (t, s) \in T, g \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P}), \quad (2.4)$$

3. *exponentially unstable in $\varkappa$ -mean* if there exist positive constants M, ε such that

$$\mathbb{E}_\varkappa[\Phi(t, s, \cdot)g(\cdot)] \geq M e^{\varepsilon(t-s)} \mathbb{E}_\varkappa[g], \quad \forall (t, s) \in T, g \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P}), \quad (2.5)$$

4. *injective in the stochastic sense* if for every $(p, q) \in T$ and $g \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[g] \neq 0$ we have

$$\mathbb{E}_\varkappa[\Phi(p, q, \cdot)g(\cdot)] > 0. \quad (2.6)$$

Remark 2.5. It is elementary to check that if a stochastic skew-evolution semiflow is exponentially stable (unstable, bounded) in the classical sense, then it must be exponentially stable (unstable, bounded, respectively) in the stochastic sense. But the inverse direction fails to hold. To give an example for this claim, we consider the case when $\mathcal{M}$ has a partition $\{\mathcal{M}_0, \dots, \mathcal{M}_\kappa\}$, where $\mathcal{M}_0 \neq \emptyset$ and $\mu(\mathcal{M}_0) = 0$. Assume that for each $n \in \{1, \dots, \kappa\}$ there exist positive constants K_n, ε_n such that for every $(p, q) \in T, g \in \mathcal{L}^1(\mathcal{M}, \mathbf{P})$ we have

$$\int_{\mathcal{M}_n} \|\Phi(p, q, \omega)g(\omega)\| d\mathbf{P}(\omega) \leq K_n e^{-\varepsilon_n(p-q)} \int_{\mathcal{M}_n} \|g(\omega)\| d\mathbf{P}(\omega).$$

Furthermore, assume that there exists a positive constant K_0 such that for every $(p, q) \in T, g \in \mathcal{L}^1(\mathcal{M}, \mathbf{P})$ we have

$$\int_{\mathcal{M}_0} \|\Phi(p, q, \omega)g(\omega)\| d\mathbf{P}(\omega) \leq K_0 \int_{\mathcal{M}_0} \|g(\omega)\| d\mathbf{P}(\omega).$$

Under these assumptions, (ζ, Φ) is exponentially stable in mean with $\varkappa = 1$, but it is not exponentially stable.

2.3. BANACH SPACES OF FUNCTIONS OR SEQUENCES

The symbol (Ω, Σ, μ) denotes a positive σ -finite measure space, and $S_\mu(\Sigma, \mathbb{C})$ refers to the linear space of μ -measurable functions $f : \Omega \rightarrow \mathbb{C}$. For $f, g \in S_\mu(\Sigma, \mathbb{C})$, writing $f = g$ means that they are μ -almost everywhere.

Definition 2.6. A map $\Xi : S_\mu(\Sigma, \mathbb{C}) \rightarrow [0, \infty]$ is called a *Banach function norm* if the following conditions hold:

1. $\Xi(zg) = |z|\Xi(g)$, $\forall z \in \mathbb{C}$, $\forall g \in S_\mu(\Sigma, \mathbb{C})$ with $\Xi(g) < \infty$,
2. if $|g| \leq |h|$ μ -almost everywhere, then $\Xi(g) \leq \Xi(h)$,
3. $\Xi(g) = 0$ if and only if $g = 0$ μ -almost everywhere,
4. $\Xi(g + h) \leq \Xi(g) + \Xi(h)$, $\forall g, h \in S_\mu(\Sigma, \mathbb{C})$.

The normed linear space $\Delta := \{f \in S_\mu(\Sigma, \mathbb{C}) : \Xi(f) < \infty\}$ is called a *Banach function space* if it is complete with respect to the norm $|f|_\Delta := \Xi(f)$.

The reader can refer to the monograph [10] for details on the theory of Banach function spaces. We focus on two cases as $(\mathbb{Z}_{\geq 0}, \mathcal{P}(\mathbb{Z}_{\geq 0}), \mu_c)$ and $(\mathbb{R}, \sigma(\mathbb{R}), m)$, where μ_c is the counting measure and m is the Lebesgue measure.

In the process of generalizing Datko's theorem, Megan *et al.* [8] gave the following spaces.

The symbol $\mathcal{B}(\mathbb{Z}_{\geq 0})$ refers to the set of all Banach sequence spaces $\mathcal{T}$ satisfying both following conditions:

$$\inf_{p \in \mathbb{Z}_{\geq 0}} |\mathcal{X}_{\{p\}}|_{\mathcal{T}} > 0, \quad (2.7)$$

$$\lim_{q \rightarrow \infty} |\mathcal{X}_{\{0, \dots, q\}}|_{\mathcal{T}} = \infty. \quad (2.8)$$

$\mathcal{B}(\mathbb{R})$ denotes the set of all Banach function spaces $\mathcal{E}$ with the property that $\lim_{t \rightarrow \infty} |\mathcal{X}_{[0, t)}|_{\mathcal{E}} = \infty$ and there is a strictly increasing sequence $\{s_n\}$ of positive real numbers with

$$\lim_{n \rightarrow \infty} s_n = \infty, \quad \sup(s_{n+1} - s_n) < \infty, \quad \inf_n |\mathcal{X}_{[s_n, s_{n+1})}|_{\mathcal{E}} > 0.$$

In [7], the author showed that $\mathcal{B}(\mathbb{Z}_{\geq 0})$ is a particular case of the following.

The set $\mathcal{H}(\mathbb{Z}_{\geq 0})$ contains all Banach sequence spaces $\mathcal{T}$ such that:

1. there are $a_1, b_1 > 0$, $\{\varsigma(m)\} \in \mathcal{S}(a_1, b_1)$, and a finite set $E \subset \mathbb{Z}_{\geq 0}$ with the property

$$\varsigma : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}, \quad \varsigma(p) \geq 1, \forall p \geq 1, \quad \inf_{p \in \mathbb{Z}_{\geq 0}} |\mathcal{X}_{\{\varsigma(p)+q : q \in E\}}|_{\mathcal{T}} > 0, \quad (2.9)$$

2. there is $\{n_d\} \subset \mathbb{Z}_{\geq 0}$ with the following property

$$\lim_{d \rightarrow \infty} n_d = \infty, \quad \lim_{d \rightarrow \infty} |\mathcal{X}_{\{n_0, \dots, n_d\}}|_{\mathcal{T}} = \infty. \quad (2.10)$$

In [7], the author succeeded to characterize a *polynomial stability* via the existence of spaces in $\mathcal{H}(\mathbb{Z}_{\geq 0})$.

3. STABILITY IN $\varkappa$ -MEAN

In this section, we point out versions of Datko's theorem for the stability in the stochastic sense. More precisely, we give characterizations for the exponential stability in $\varkappa$ -mean of stochastic skew-evolution semiflows, in terms of Banach spaces of functions or sequences.

3.1. SOME INITIAL PROPERTIES

We prove two observations that will be needed later on. In Lemma 3.1, we establish a relationship between the exponential stability in $\varkappa$ -mean and contraction operators in the stochastic sense. Lemma 3.2 isolates a sufficient condition for a stochastic skew-evolution semiflow to be bounded in the stochastic sense.

Despite the possible familiarity of Lemma 3.1, we give a proof to present a complete and thorough exposition.

Lemma 3.1. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially stable in $\varkappa$ -mean if and only if there exist $d \in [0, 1)$, $\lambda \in \mathbb{Z}_{\geq 1}$, such that the inequality*

$$\mathbb{E}_{\varkappa}[\Phi(\lambda + m, m, \cdot)u(\cdot)] \leq d\mathbb{E}_{\varkappa}[u]$$

holds for every $m \in \mathbb{Z}_{\geq 0}$, $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$.

Proof. The necessity is left to the reader as it is simple. We prove the sufficiency as follows. Arbitrarily take $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$. By the arguments of the induction on k , we can show that

$$\mathbb{E}_{\varkappa}[\Phi(k\lambda + m, m, \cdot)u(\cdot)] \leq d^k \mathbb{E}_{\varkappa}[u], \quad \forall m \in \mathbb{Z}_{\geq 0}, \forall k \in \mathbb{Z}_{\geq 0}.$$

Let $t \geq 0$. Then we can write $t = \lambda p + r$, where $p \in \mathbb{Z}_{\geq 0}$ and $r \in [0, \lambda)$. By (2.3), we estimate

$$\begin{aligned} \mathbb{E}_{\varkappa}[\Phi(t + m, m, \cdot)u(\cdot)] &\leq Me^{\varepsilon\lambda} \mathbb{E}_{\varkappa}[\Phi(\lambda p + m, m, \cdot)u(\cdot)] \\ &\leq Me^{\varepsilon\lambda} d^p \mathbb{E}_{\varkappa}[u] \\ &\leq K_1 e^{-\alpha t} \mathbb{E}_{\varkappa}[u], \end{aligned}$$

where $K_1 = Me^{\varepsilon\lambda}d^{-1}$, $\alpha = -\lambda^{-1} \ln d$, and $p > t\lambda^{-1} - 1$. Let $s \geq 0$. There are two possibilities as follows.

- If $t + s \leq [s] + 1$, then by (2.3) we see

$$\mathbb{E}_{\varkappa}[\Phi(t + s, s, \cdot)u(\cdot)] \leq Me^{\varepsilon} \mathbb{E}_{\varkappa}[u] \leq Me^{\varepsilon+\alpha} e^{-\alpha t} \mathbb{E}_{\varkappa}[u].$$

- If $t + s \geq [s] + 1$, then by (3.1) we get

$$\begin{aligned} \mathbb{E}_{\varkappa}[\Phi(t + s, s, \cdot)u(\cdot)] &\leq K_1 e^{-\alpha(t+s-[s]-1)} \mathbb{E}_{\varkappa}[\Phi([s] + 1, s, \cdot)u(\cdot)] \\ &\leq K_1 e^{-\alpha(t+s-[s]-1)} Me^{\varepsilon} \mathbb{E}_{\varkappa}[u] \\ &\leq K_1 Me^{\alpha+\varepsilon} e^{-\alpha t} \mathbb{E}_{\varkappa}[u], \end{aligned}$$

where the second inequality uses (2.3).

Thus, we conclude that

$$\mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)] \leq \max\{K_1 M e^{\alpha+\varepsilon}, M e^{\varepsilon+\alpha}\} e^{-\alpha t} \mathbb{E}_\varkappa[u]. \quad \square$$

Lemma 3.2. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Assume that there exist positive constants δ, L , an above unbounded sequence $\{t_n\} \in \mathcal{S}(\delta)$ such that the inequality*

$$\mathbb{E}_\varkappa[\Phi(t_n + m, m, \cdot)u(\cdot)] \leq L \mathbb{E}_\varkappa[u]$$

holds for every $m, n \in \mathbb{Z}_{\geq 0}$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$. Then there exists $K > 0$ such that the inequality

$$\mathbb{E}_\varkappa[\Phi(p+q, q, \cdot)u(\cdot)] \leq K \mathbb{E}_\varkappa[u]$$

holds for every $p, q \geq 0$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$.

Proof. Let $m \in \mathbb{Z}_{\geq 0}$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$, and $t \in \mathbb{R}_{\geq 0}$. Since $\{t_n\} \in \mathcal{S}(\delta)$, there exists n such that $t_n \leq t < t_{n+1}$. By (2.3), we estimate

$$\mathbb{E}_\varkappa[\Phi(t+m, m, \cdot)u(\cdot)] \leq M e^{\varepsilon(t-t_n)} \mathbb{E}_\varkappa[\Phi(t_n+m, m, \cdot)u(\cdot)] \leq M e^{\varepsilon\delta} L \mathbb{E}_\varkappa[u],$$

where the last inequality employs the fact that $t_{n+1} - t_n \leq \delta$.

Let $s \geq 0$. There are two cases as follows.

- If $t+s \leq [s] + 1$, then by (2.3) we see

$$\mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)] \leq M e^\varepsilon \mathbb{E}_\varkappa[u].$$

- If $t+s > [s] + 1$, then

$$\mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)] \leq M e^{\varepsilon\delta} L \mathbb{E}_\varkappa[\Phi([s]+1, s, \cdot)u(\cdot)] \leq M^2 e^{\varepsilon\delta+\varepsilon} L \mathbb{E}_\varkappa[u].$$

Thus, we can chose $K = \max\{M e^\varepsilon, M^2 e^{\varepsilon\delta+\varepsilon} L\}$. $\square$

3.2. DISCRETE VERSION

With all preparation in place, we now are ready to state and prove the first main result. A discrete-time version of Datko's theorem for the exponential stability in $\varkappa$ -mean is given by the result below.

Theorem 3.3. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially stable in $\varkappa$ -mean if and only if there exist $K, \delta > 0$, $\mathcal{E} \in \mathcal{H}(\mathbb{Z}_{\geq 0})$, $\vartheta \in F$, and $\{t_n\} \in \mathcal{S}(\delta)$ such that:*

1. *for every $(z, u) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$, the sequence $f_{z,u} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$*

$$f_{z,u}(j) = \vartheta(\mathbb{E}_\varkappa[\Phi(t_j + z, z, \cdot)u(\cdot)])$$

belongs to $\mathcal{E}$,

2. *the inequality $|f_{z,u}|_\mathcal{E} \leq K$ holds for every $z \in \mathbb{R}_{\geq 0}$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[u] = 1$.*

Proof. Necessity. Suppose that (ζ, Φ) is exponentially stable in $\varkappa$ -mean. Then (2.4) holds, and so

$$\sum_{j \geq 0} \mathbb{E}_{\varkappa}[\Phi(j + z, z, \cdot)u(\cdot)] \leq \sum_{j \geq 0} M e^{-\varepsilon j} < \infty.$$

Thus, for taking $t_j = j$, $\vartheta(t) = t$ and $\mathcal{E} = \ell^1(\mathbb{Z}_{\geq 0})$, the necessary condition follows.

Sufficiency. Let $z \in \mathbb{R}_{\geq 0}$ and $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] = 1$.

Case 1. $q = \sup\{t_n : n \in \mathbb{Z}_{\geq 0}\} < \infty$.

For every $j \in \{n_0, \dots, n_k\}$, by (2.3), we have

$$\mathbb{E}_{\varkappa}[\Phi(q + z, z, \cdot)u(\cdot)] \leq M e^{\varepsilon q} \mathbb{E}_{\varkappa}[\Phi(t_j + z, z, \cdot)u(\cdot)],$$

which gives

$$f_{z,u} \geq \vartheta \left(\frac{1}{M e^{\varepsilon q}} \mathbb{E}_{\varkappa}[\Phi(q + z, z, \cdot)u(\cdot)] \right) \mathcal{X}_{\{n_0, \dots, n_k\}}.$$

We infer that

$$K \geq |f_{z,u}|_{\mathcal{E}} \geq \vartheta \left(\frac{1}{M e^{\varepsilon q}} \mathbb{E}_{\varkappa}[\Phi(q + z, z, \cdot)u(\cdot)] \right) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_{\mathcal{E}},$$

which implies, by (2.10), that $\mathbb{E}_{\varkappa}[\Phi(q + z, z, \cdot)u(\cdot)] = 0$. Hence by Lemma 3.1, (ζ, Φ) is exponentially stable in $\varkappa$ -mean.

Case 2. $\sup\{t_n : n \in \mathbb{Z}_{\geq 0}\} = \infty$.

In view of (2.9), there exist $a_1, b_1 > 0$, $\{\varsigma(q)\} \in \mathcal{S}(a_1, b_1)$, and a finite set $E \subset \mathbb{Z}_{\geq 0}$ such that

$$\varsigma(q) \geq 1, \forall q \geq 1, \quad \inf_{q \in \mathbb{Z}_{\geq 0}} |\mathcal{X}_{\{\varsigma(q)+k: k \in E\}}|_{\mathcal{E}} > 0.$$

Setting

$$p = \max E, \quad r = \inf_{q \in \mathbb{Z}_{\geq 0}} |\mathcal{X}_{\{\varsigma(q)+k: k \in E\}}|_{\mathcal{E}}.$$

Let $m \in \mathbb{Z}_{\geq 0}$. There are two possibilities as follows.

– If $\varsigma(m) \leq p$, then by (2.3) we see

$$\mathbb{E}_{\varkappa}[\Phi(t_{\varsigma(m)} + z, z, \cdot)u(\cdot)] \leq M e^{\varepsilon t_p} \mathbb{E}_{\varkappa}[u].$$

– If $\varsigma(m) \geq p + 1$, then for $j \in \{\varsigma(m) + k : k \in E\}$, by (2.3) the following is obtained

$$\begin{aligned} \mathbb{E}_{\varkappa}[\Phi(t_{\varsigma(m)+p} + z, z, \cdot)u(\cdot)] &\leq M e^{\varepsilon(t_{\varsigma(m)+p} - t_j)} \mathbb{E}_{\varkappa}[\Phi(t_j + z, z, \cdot)u(\cdot)] \\ &\leq M e^{\varepsilon \delta(\varsigma(m)+p-j)} \mathbb{E}_{\varkappa}[\Phi(t_j + z, z, \cdot)u(\cdot)] \\ &\leq M e^{\varepsilon \delta p} \mathbb{E}_{\varkappa}[\Phi(t_j + z, z, \cdot)u(\cdot)], \end{aligned}$$

and so

$$f_{z,u} \geq \vartheta \left(\frac{1}{M e^{\varepsilon \delta p}} \mathbb{E}_{\varkappa}[\Phi(t_{\varsigma(m)+p} + z, z, \cdot)u(\cdot)] \right) \mathcal{X}_{\{\varsigma(m)+k: k \in E\}}.$$

By the assumption of K , we estimate

$$\begin{aligned} K &\geq |f_{z,u}|_{\mathcal{E}} \geq \vartheta \left(\frac{1}{Me^{\varepsilon\delta p}} \mathbb{E}_{\mathcal{K}}[\Phi(t_{\zeta(m)+p} + z, z, \cdot)u(\cdot)] \right) |\mathcal{X}_{\{\zeta(m)+k: k \in E\}}|_{\mathcal{E}} \\ &\geq r\vartheta \left(\frac{1}{Me^{\varepsilon\delta p}} \mathbb{E}_{\mathcal{K}}[\Phi(t_{\zeta(m)+p} + z, z, \cdot)u(\cdot)] \right), \end{aligned}$$

which implies, as $\lim_{t \rightarrow \infty} \vartheta(t) = \infty$, that there exists a constant $K_1 > 0$ with

$$\mathbb{E}_{\mathcal{K}}[\Phi(t_{\zeta(m)+p} + z, z, \cdot)u(\cdot)] \leq K_1.$$

Hence, by Lemma 3.2 there exists a positive constant K_2 such that the inequality

$$\mathbb{E}_{\mathcal{K}}[\Phi(t + s, s, \cdot)u(\cdot)] \leq K_2 \mathbb{E}_{\mathcal{K}}[u]$$

holds for every $t, s \geq 0$, $u \in \mathcal{L}^{\mathcal{K}}(\mathcal{M}, \mathbb{X})$.

Recall that the sequence $\{n_k\}$ satisfies (2.10). For every $j \in \{n_0, \dots, n_k\}$, we have

$$\mathbb{E}_{\mathcal{K}}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)] \leq K_2 \mathbb{E}_{\mathcal{K}}[\Phi(t_j + z, z, \cdot)u(\cdot)],$$

and so

$$f_{z,u} \geq \vartheta \left(\frac{1}{K_2} \mathbb{E}_{\mathcal{K}}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)] \right) |\mathcal{X}_{\{n_0, \dots, n_k\}}|.$$

It follows from the assumption of K , that

$$K \geq |f_{z,u}|_{\mathcal{E}} \geq \vartheta \left(\frac{1}{K_2} \mathbb{E}_{\mathcal{K}}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)] \right) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_{\mathcal{E}}.$$

On the other hand, by (2.3), we estimate

$$\mathbb{E}_{\mathcal{K}}[\Phi(\lfloor t_{n_k} \rfloor + 1 + z, z, \cdot)u(\cdot)] \leq Me^{\varepsilon} \mathbb{E}_{\mathcal{K}}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)].$$

Thus, we get

$$\vartheta \left(\frac{1}{Me^{\varepsilon} K_2} \mathbb{E}_{\mathcal{K}}[\Phi(\lfloor t_{n_k} \rfloor + 1 + z, z, \cdot)u(\cdot)] \right) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_{\mathcal{E}} \leq K.$$

By (2.10) we can choose k with $|\mathcal{X}_{\{n_0, \dots, n_k\}}|_{\mathcal{E}} \geq K\vartheta \left(\frac{1}{2Me^{\varepsilon} K_2} \right)^{-1}$, and so by Lemma 3.1 we get the desired conclusion. $\square$

The below result is a consequence of Theorem 3.3 by taking $\mathcal{E} = \ell^1(\mathbb{Z}_{\geq 0})$.

Corollary 3.4. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\mathcal{K} \geq 1$. Then it is exponentially stable in $\mathcal{K}$ -mean if and only if there exist a function $\vartheta \in F$, and a positive constant K such that the inequality*

$$\sum_{j=0}^{\infty} \vartheta(\mathbb{E}_{\mathcal{K}}[\Phi(j + z, z, \cdot)g(\cdot)]) \leq K$$

holds for every $z \geq 0$, $g \in \mathcal{L}^{\mathcal{K}}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\mathcal{K}}[g] = 1$.

3.3. CONTINUOUS VERSION

In this subsection, we obtain continuous versions of the famous theorem of Datko for the notion of exponential stability in mean. Our proof makes use of Theorem 3.3.

Theorem 3.5. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially stable in $\varkappa$ -mean if and only if there exist $K > 0$, $\mathcal{Q} \in \mathcal{B}(\mathbb{R}_{\geq 0})$, and $\xi \in F$ such that:*

1. *for every $(z, u) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$, the function $g_{z,u} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$*

$$g_{z,u}(t) = \xi(\mathbb{E}_\varkappa[\Phi(t+z, z, \cdot)u(\cdot)])$$

belongs to $\mathcal{Q}$,

2. *the inequality $|g_{z,u}|_{\mathcal{Q}} \leq K$ holds for every $z \in \mathbb{R}_{\geq 0}, u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[u] = 1$.*

Proof. Necessity. It is immediate by taking $\xi(t) = t$ and $\mathcal{Q} = L^1(\mathbb{R}_{\geq 0})$.

Sufficiency. Since $\mathcal{Q} \in \mathcal{B}(\mathbb{R}_{\geq 0})$, there exists a strictly increasing sequence $\{s_n\}$ of positive real numbers with

$$\lim_{n \rightarrow \infty} s_n = \infty, \quad \delta = \sup(s_{n+1} - s_n) < \infty, \quad \inf_n |\mathcal{X}_{[s_n, s_{n+1})}|_{\mathcal{Q}} > 0.$$

For every $(z, u) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$, let us define the sequence $f_{z,u} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by setting

$$f_{z,u}(j) = \vartheta(\mathbb{E}_\varkappa[\Phi(s_{j+1} + z, z, \cdot)u(\cdot)]),$$

where $\vartheta(t) = \xi(\frac{t}{Me^{\varepsilon\delta}})$.

Let $s \geq 0$. Then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $s \in [s_k, s_{k+1})$. Thus, by (2.3), we have

$$\begin{aligned} \mathbb{E}_\varkappa[\Phi(s_{k+1} + z, z, \cdot)u(\cdot)] &\leq Me^{\varepsilon(s_{k+1}-s)} \mathbb{E}_\varkappa[\Phi(s + z, z, \cdot)u(\cdot)] \\ &\leq Me^{\varepsilon\delta} \mathbb{E}_\varkappa[\Phi(s + z, z, \cdot)u(\cdot)], \end{aligned}$$

which gives $g_{z,u} \geq \sum_{k \geq 0} f_{z,u}(k) \mathcal{X}_{[k, k+1)}$, and so

$$|g_{z,u}|_{\mathcal{Q}} \geq \left| \sum_{k \geq 0} f_{z,u}(k) \mathcal{X}_{[k, k+1)} \right|_{\mathcal{Q}}. \quad (3.1)$$

The right-hand side expression leads to consider the Banach sequence space

$$S_{\mathcal{Q}} = \left\{ \{\alpha(n)\}_{n \geq 0} : \sum_{n \geq 0} \alpha(n) \mathcal{X}_{[n, n+1)} \in \mathcal{Q} \right\} \quad (3.2)$$

with respect to the norm

$$|\{\alpha(n)\}_{n \in \mathbb{Z}_{\geq 0}}|_{S_{\mathcal{Q}}} := \left| \sum_{n \geq 0} \alpha(n) \mathcal{X}_{[n, n+1)} \right|_{\mathcal{Q}}. \quad (3.3)$$

It can be found that $S_{\mathcal{Q}} \in \mathcal{B}(\mathbb{Z}_{\geq 0}) \subset \mathcal{H}(\mathbb{Z}_{\geq 0})$. Inequality (3.1) and Theorem 3.3 show that (ζ, Φ) is exponentially stable in $\varkappa$ -mean. $\square$

The following result is a consequence of Theorem 3.5 by taking $\mathcal{Q} = L^1(\mathbb{R}_{\geq 0})$.

Corollary 3.6. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially stable in $\varkappa$ -mean if and only if there exist a function $\xi \in F$, and a positive constant K such that the inequality*

$$\int_0^\infty \xi(\mathbb{E}_\varkappa[\Phi(t+z, z, \cdot)g(\cdot)]) dt \leq K$$

holds for every $z \geq 0$, $g \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[g] = 1$.

4. INSTABILITY IN $\varkappa$ -MEAN

In this section, we extend the well-known Datko's theorem to the case of *instability*. Using Banach spaces of functions or sequences, we give necessary and sufficient conditions for the exponential instability in $\varkappa$ -mean of stochastic skew-evolution semiflows. Like as the stability, we also consider two cases, such as: discrete time and continuous time.

4.1. SOME INITIAL PROPERTIES

Extending the techniques used in Lemmas 3.1-3.2, we obtain the following observations.

Lemma 4.1 might be standard, but we provide its proof for the sake of completeness.

Lemma 4.1. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially unstable in $\varkappa$ -mean if and only if there exist $d > 1$, $\lambda \in \mathbb{Z}_{\geq 1}$, such that the inequality*

$$\mathbb{E}_\varkappa[\Phi(\lambda + s, s, \cdot)u(\cdot)] \geq d\mathbb{E}_\varkappa[u]$$

holds for every $s \in \mathbb{R}_{\geq 0}, u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$.

Proof. The necessity is left to the reader as it is simple. We prove the sufficiency as follows. Arbitrarily take $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$. By the arguments of the induction on k , we can show that

$$\mathbb{E}_\varkappa[\Phi(k\lambda + s, s, \cdot)u(\cdot)] \geq d^k \mathbb{E}_\varkappa[u], \quad \forall s \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}.$$

Let $t \geq 0$. Then we can write $t = \lambda p + r$, where $p \in \mathbb{Z}_{\geq 0}$ and $r \in [0, \lambda)$. By (2.3), we estimate

$$\begin{aligned} \mathbb{E}_\varkappa[\Phi((p+1)\lambda + s, s, \cdot)u(\cdot)] &\leq M e^{\varepsilon(\lambda-r)} \mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)] \\ &\leq M e^{\varepsilon\lambda} \mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)], \end{aligned}$$

which gives

$$\begin{aligned} \mathbb{E}_\varkappa[\Phi(t+s, s, \cdot)u(\cdot)] &\geq M^{-1} e^{-\varepsilon\lambda} \mathbb{E}_\varkappa[\Phi((p+1)\lambda + s, s, \cdot)u(\cdot)] \\ &\geq M^{-1} e^{-\varepsilon\lambda} d^{p+1} \mathbb{E}_\varkappa[u] \geq K e^{\alpha t} \mathbb{E}_\varkappa[u], \end{aligned}$$

where $K = M^{-1} e^{-\varepsilon\lambda}$, $\alpha = \lambda^{-1} \ln d$, and $p+1 > \frac{t}{\lambda}$. □

Lemma 4.2. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Assume that there exist two positive constants δ, L , an above unbounded sequence $\{t_n\} \in \mathcal{S}(\delta)$ such that*

$$\mathbb{E}_\varkappa[\Phi(t_n + s, s, \cdot)u(\cdot)] \geq L\mathbb{E}_\varkappa[u], \quad \forall n \in \mathbb{Z}_{\geq 0}, s \geq 0, u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P}).$$

Then there exists $Q > 0$ such that the inequality

$$\mathbb{E}_\varkappa[\Phi(p + q, q, \cdot)u(\cdot)] \geq Q\mathbb{E}_\varkappa[u] \quad (4.1)$$

holds for every $p, q \geq 0$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$.

Proof. Let $s \geq 0$, $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$, and $t \in \mathbb{R}_{\geq 0}$. Since $\{t_n\} \in \mathcal{S}(\delta)$, there exists n such that $t_n \leq t < t_{n+1}$. The assumption of L and (2.3) give

$$L\mathbb{E}_\varkappa[u] \leq \mathbb{E}_\varkappa[\Phi(t_{n+1} + s, s, \cdot)u(\cdot)] \leq Me^{\varepsilon\delta}\mathbb{E}_\varkappa[\Phi(t + s, s, \cdot)u(\cdot)],$$

where the last inequality employs the fact that $t_{n+1} - t_n \leq \delta$. $\square$

4.2. DISCRETE VERSION

Armed with Lemmas 4.1–4.2, we are ready to prove a discrete version of Datko's theorem for the concept of exponential *instability* in mean.

Theorem 4.3. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially unstable in $\varkappa$ -mean if and only if it is injective in the stochastic sense (that is (2.6)) and there exist $K, \delta > 0$, $\mathcal{E} \in \mathcal{H}(\mathbb{Z}_{\geq 0})$, $\Lambda \in \mathcal{F}$, and $\{t_n\} \in \mathcal{S}(\delta)$ such that*

1. *for every $(z, u) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[u] > 0$, the sequence $F_{z,u} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$*

$$F_{z,u}(j) = \Lambda(\mathbb{E}_\varkappa[\Phi(t_j + z, z, \cdot)u(\cdot)])$$

belongs to $\mathcal{E}$.

2. *the inequality $|F_{z,u}|_\mathcal{E} \leq K$ holds for every $z \in \mathbb{R}_{\geq 0}, u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[u] = 1$.*

Proof. Necessity. Suppose that (ζ, Φ) is exponentially unstable in $\varkappa$ -mean. Then (2.5) holds, and so

$$\sum_{j \geq 0} \mathbb{E}_\varkappa[\Phi(j + z, z, \cdot)u(\cdot)]^{-1} \leq \sum_{j \geq 0} M^{-1}e^{-\varepsilon j} < \infty.$$

Thus, for taking $t_j = j$, $\Lambda(t) = \frac{1}{t}$ and $\mathcal{E} = \ell^1(\mathbb{Z}_{\geq 0})$, the necessary condition follows.

Sufficiency. Let $z \in \mathbb{R}_{\geq 0}$ and $u \in \mathcal{L}^\varkappa(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_\varkappa[u] = 1$. By (2.10), there exists $\{n_k\} \subset \mathbb{Z}_{\geq 0}$ such that

$$\lim_{k \rightarrow \infty} n_k = \infty, \quad \lim_{k \rightarrow \infty} |\mathcal{X}_{\{n_0, \dots, n_k\}}|_\mathcal{E} = \infty.$$

Using a contradiction argument, we show that $\{t_j\}$ is above unbounded. Indeed, assume that $p := \sup\{t_j : j \in \mathbb{Z}_{\geq 0}\} < \infty$. For $j \in \{n_0, \dots, n_k\}$, by (2.3) we have

$$\mathbb{E}_\varkappa[\Phi(t_j + z, z, \cdot)u(\cdot)] \leq Me^{p\varepsilon},$$

which gives

$$F_{z,u} \geq \Lambda(Me^{p^\varepsilon}) \mathcal{X}_{\{n_0, \dots, n_k\}}$$

and hence we can estimate

$$K \geq |F_{z,u}|_\mathcal{E} \geq \Lambda(Me^{p^\varepsilon}) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_\mathcal{E};$$

but this is impossible.

Note that by (2.9), there exist $a_1, b_1 > 0$, $\{\varsigma(q)\} \in \mathcal{S}(a_1, b_1)$, and a finite set $E \subset \mathbb{Z}_{\geq 0}$ such that

$$\varsigma(q) \geq 1, \forall q \geq 1, \quad \eta = \inf_{q \in \mathbb{Z}_{\geq 0}} |\mathcal{X}_{\{\varsigma(q)+k: k \in E\}}|_\mathcal{E} > 0.$$

Let $\ell = \max E$ and $n \in \mathbb{Z}_{\geq 1}$. For every $j \in \{\varsigma(n) + k : k \in E\}$, by (2.3) we have

$$\begin{aligned} \mathbb{E}_\mathcal{X}[\Phi(t_j + z, z, \cdot)u(\cdot)] &\leq Me^{\varepsilon\delta(j-\varsigma(n))} \mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)] \\ &\leq Me^{\varepsilon\delta\ell} \mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)]. \end{aligned}$$

Consequently,

$$F_{z,u} \geq \Lambda(Me^{\varepsilon\delta\ell} \mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)]) \mathcal{X}_{\{\varsigma(n)+k: k \in E\}}.$$

By the assumption of K , we infer that

$$\begin{aligned} K &\geq \Lambda(Me^{\varepsilon\delta\ell} \mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)]) \cdot |\mathcal{X}_{\{\varsigma(n)+k: k \in E\}}|_\mathcal{E} \\ &\geq \eta \Lambda(Me^{\varepsilon\delta\ell} \mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)]). \end{aligned}$$

Since $\lim_{t \rightarrow 0} \Lambda(t) = \infty$, there exists a positive constant K_1 such that

$$\mathbb{E}_\mathcal{X}[\Phi(t_{\varsigma(n)} + z, z, \cdot)u(\cdot)] \geq K_1,$$

and hence by Lemma 4.2 there exists a positive constant Q such that (4.1) holds. For every $j \in \{n_0, \dots, n_k\}$, we have

$$\mathbb{E}_\mathcal{X}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)] \geq Q \mathbb{E}_\mathcal{X}[\Phi(t_j + z, z, \cdot)u(\cdot)],$$

which gives

$$F_{z,u} \geq \Lambda(Q^{-1} \mathbb{E}_\mathcal{X}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)]) \mathcal{X}_{\{n_0, \dots, n_k\}}.$$

Since $|F_{z,u}|_\mathcal{E} \leq K$, we see that

$$K \geq \Lambda(Q^{-1} \mathbb{E}_\mathcal{X}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)]) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_\mathcal{E}.$$

Note that by (2.3),

$$\mathbb{E}_\mathcal{X}[\Phi(t_{n_k} + z, z, \cdot)u(\cdot)] \leq Me^\varepsilon \mathbb{E}_\mathcal{X}[\Phi(\lfloor t_{n_k} \rfloor + z, z, \cdot)u(\cdot)].$$

Thus, we get

$$K \geq \Lambda(Q^{-1} Me^\varepsilon \mathbb{E}_\mathcal{X}[\Phi(\lfloor t_{n_k} \rfloor + z, z, \cdot)u(\cdot)]) \cdot |\mathcal{X}_{\{n_0, \dots, n_k\}}|_\mathcal{E}.$$

For choosing n_k with $|\mathcal{X}_{\{n_0, \dots, n_k\}}|_{\mathcal{E}} \geq K\Lambda(2Q^{-1}Me^{\varepsilon})^{-1}$, it follows from the above inequality, that

$$\mathbb{E}_{\varkappa}[\Phi(\lfloor t_{n_k} \rfloor + z, z, \cdot)u(\cdot)] \geq 2,$$

and hence by Lemma 4.1 we get the desired conclusion. $\square$

The following result can be easily proved by using Theorem 4.3 for $\mathcal{E} = \ell^1(\mathbb{Z}_{\geq 0})$.

Corollary 4.4. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially unstable in $\varkappa$ -mean if and only if it is injective in the stochastic sense and there exist $\Lambda \in \mathcal{F}$ and $K > 0$ such that the inequality*

$$\sum_{j=0}^{\infty} \Lambda(\mathbb{E}_{\varkappa}[\Phi(j + z, z, \cdot)u(\cdot)]) \leq K$$

for every $z \geq 0$, $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] = 1$.

4.3. CONTINUOUS TIME

Having obtained discrete-time characterization, we give a continuous-time version for the exponential instability in $\varkappa$ -mean, by means of Banach function spaces.

Theorem 4.5. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially unstable in $\varkappa$ -mean if and only if it is injective in the stochastic sense (that is (2.6)) and there exist $K > 0$, $\mathcal{Q} \in \mathcal{B}(\mathbb{R}_{\geq 0})$ and $\Gamma \in \mathcal{F}$, such that:*

1. for every $(z, u) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] > 0$, the function $G_{z,u} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$

$$G_{z,u}(t) = \Gamma(\mathbb{E}_{\varkappa}[\Phi(t + z, z, \cdot)u(\cdot)])$$

belongs to $\mathcal{Q}$,

2. the inequality $|G_{z,u}|_{\mathcal{Q}} \leq K$ holds for every $z \in \mathbb{R}_{\geq 0}$, $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] = 1$.

Proof. Necessity. It is immediate by taking $\mathcal{Q} = L^1(\mathbb{R}_{\geq 0})$ and $\Gamma(t) = \frac{1}{t}$.

Sufficiency. Since $\mathcal{Q} \in \mathcal{B}(\mathbb{R}_{\geq 0})$, there exists a strictly increasing sequence $\{s_n\}$ of positive real numbers with

$$\lim_{n \rightarrow \infty} s_n = \infty, \quad \delta = \sup(s_{n+1} - s_n) < \infty, \quad \inf_n |\mathcal{X}_{[s_n, s_{n+1})}|_{\mathcal{Q}} > 0.$$

For every $(z, g) \in \mathbb{R}_{\geq 0} \times \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] > 0$, let us define the sequence $F_{z,u} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by setting

$$F_{z,u}(j) = \Lambda(\mathbb{E}_{\varkappa}[\Phi(s_j + z, z, \cdot)u(\cdot)]),$$

where $\Lambda(t) = \Gamma(Me^{\varepsilon\delta}t)$.

Let $s \geq 0$. Then there exists $k \in \mathbb{Z}_{\geq 0}$ such that $s \in [s_k, s_{k+1})$. Thus, by (2.3), we have

$$\begin{aligned} \mathbb{E}_{\varkappa}[\Phi(s + z, z, \cdot)u(\cdot)] &\leq Me^{\varepsilon(s-s_k)} \mathbb{E}_{\varkappa}[\Phi(s_k + z, z, \cdot)u(\cdot)] \\ &\leq Me^{\varepsilon\delta} \mathbb{E}_{\varkappa}[\Phi(s_k + z, z, \cdot)u(\cdot)], \end{aligned}$$

which gives $G_{z,u} \geq \sum_{k \geq 0} F_{z,u}(k) \mathcal{X}_{[k,k+1]}$, and so

$$|G_{z,u}|_{\mathcal{Q}} \geq \left| \sum_{k \geq 0} F_{z,u}(k) \mathcal{X}_{[k,k+1]} \right|_{\mathcal{Q}}. \quad (4.2)$$

Like as the stability, we also consider the Banach function space $S_{\mathcal{Q}}$ in (3.2) with the norm (3.3). By (4.2), we can apply Theorem 4.3 to $S_{\mathcal{Q}}$ and conclude that (ζ, Φ) is exponentially unstable in $\varkappa$ -mean. $\square$

The following result is a consequence of Theorem 4.5 by taking $\mathcal{Q} = L^1(\mathbb{R}_{\geq 0})$.

Corollary 4.6. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa \geq 1$. Then it is exponentially unstable in $\varkappa$ -mean if and only if it is injective in the stochastic sense and there exist $\Gamma \in \mathcal{F}$, and $K > 0$, such that the inequality*

$$\int_0^\infty \Gamma(\mathbb{E}_{\varkappa}[\Phi(t+z, z, \cdot)u(\cdot)]) dt \leq K$$

holds for every $z \geq 0$, $u \in \mathcal{L}^{\varkappa}(\mathcal{M}, \mathbf{P})$ with $\mathbb{E}_{\varkappa}[u] = 1$.

5. LYAPUNOV-TYPE EQUATIONS

Throughout the section, $\mathcal{H}$ is a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and a corresponding norm $\| \cdot \|$. In this section, we look for an extension of the Lyapunov-type equation to the polynomial (in)stability. To that aim, we use some terminologies.

Definition 5.1. A map $A : T \times \mathcal{L}^2(\mathcal{M}, \mathbf{P}) \rightarrow B(\mathcal{H})$ is called:

1. *positive* if

$$\int_{\mathcal{M}} \langle A(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \geq 0 \quad \forall (t, s) \in T, g \in \mathcal{L}^2(\mathcal{M}, \mathbf{P}),$$

2. *uniformly positive* if there exists a positive constant α such that

$$\int_{\mathcal{M}} \langle A(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \geq \alpha \mathbb{E}_2[g]^2 \quad \forall (t, s) \in T, g \in \mathcal{L}^2(\mathcal{M}, \mathbf{P}),$$

3. *uniformly negative* if $-A$ is uniformly positive,

4. *above bounded* if there exists a positive constant Δ such that

$$\int_{\mathcal{M}} \langle A(s, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \leq \Delta \mathbb{E}_2[g]^2 \quad \forall (t, s) \in T, g \in \mathcal{L}^2(\mathcal{M}, \mathbf{P}),$$

5. *below bounded* if $-A$ is above bounded.

The symbol $\mathbb{O}$ denotes the set of all uniformly positive maps. Let $\mathcal{O}_+$ be the set of all above bounded, positive maps. Let $\mathcal{O}_-$ be the set of all maps which are both below bounded and uniformly negative.

Motivated by the Lyapunov-type equation (1.2), we consider the operator equation

$$W(s, s, \omega) - \Phi^*(t, s, \omega)W(t, s, \omega)\Phi(t, s, \omega) = \int_s^t \Phi^*(\tau, s, \omega)H(\tau, s, \omega)\Phi(\tau, s, \omega) d\tau, \quad (5.1)$$

where $W, H : T \times \mathcal{M} \rightarrow B(\mathcal{H})$. In the below result, we give a relationship between the exponential stability in mean square and solvability of (5.1).

Theorem 5.2. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa = 2$. Then it is exponentially stable in mean square if there exist $W \in \mathcal{O}_+$, and $H \in \mathbb{O}$ such that (5.1) holds.*

Proof. The necessity follows by taking

$$H(t, s, \omega) = I, \quad W(t, s, \omega) = \int_t^\infty \Phi^*(\tau, t, \zeta(t, s, \omega))\Phi(\tau, t, \zeta(t, s, \omega)) d\tau.$$

We prove the sufficiency as follows. Since $W \in \mathcal{O}_+$, and $H \in \mathbb{O}$, there exist positive constants α and β such that H is uniformly positive with β and W is above bounded with α . For every $g \in \mathcal{L}^2(\mathcal{M}, \mathbf{P})$, we have that

$$\begin{aligned} \beta \mathbb{E}_2[\Phi(\tau, s, \cdot)g(\cdot)]^2 &\leq \int_{\mathcal{M}} \langle H(\tau, s, \omega)\Phi(\tau, s, \omega)g(\omega), \Phi(\tau, s, \omega)g(\omega) \rangle d\mathbf{P}(\omega) \\ &= \int_{\mathcal{M}} \langle \Phi^*(\tau, s, \omega)H(\tau, s, \omega)\Phi(\tau, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega), \end{aligned}$$

which implies, by integrating with respect to τ from s to t and using Fubini's theorem, that

$$\begin{aligned} &\beta \int_s^t \mathbb{E}_2[\Phi(\tau, s, \cdot)g(\cdot)]^2 d\tau \\ &\leq \int_{\mathcal{M}} \left\langle \int_s^t \Phi^*(\tau, s, \omega)H(\tau, s, \omega)\Phi(\tau, s, \omega)g(\omega) d\tau, g(\omega) \right\rangle d\mathbf{P}(\omega) \\ &= \int_{\mathcal{M}} \langle W(s, s, \omega)g(\omega) - \Phi^*(t, s, \omega)W(t, s, \omega)\Phi(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &\leq \int_{\mathcal{M}} \langle W(s, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P} \leq \alpha \mathbb{E}_2[g]^2. \end{aligned}$$

So, we can use Corollary 3.6 to get the desired conclusion. $\square$

In the case of instability, we also have a similar result.

Theorem 5.3. *Let (ζ, Φ) be a stochastic skew-evolution semiflow and condition (2.3) holds for $\varkappa = 2$. Then it is exponentially unstable in mean square if there exist $W \in \mathcal{O}_-$, and $H \in \mathbb{O}$ such that (5.1) holds.*

Proof. Since $W \in \mathcal{O}_-(\delta)$, and $H \in \mathbb{O}$, there exist positive constants α, β, γ such that for every $g \in \mathcal{L}^2(\mathcal{M}, \mathbf{P})$, $(t, s) \in T$

$$-\alpha \mathbb{E}_2[g]^2 \leq \int_{\mathcal{M}} \langle W(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \leq -\gamma \mathbb{E}_2[g]^2,$$

and

$$\int_{\mathcal{M}} \langle H(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \geq \beta \mathbb{E}_2[g]^2.$$

We have that

$$\begin{aligned} -\alpha \mathbb{E}_2[\Phi(t, s, \cdot)g(\cdot)]^2 &\leq \int_{\mathcal{M}} \langle \Phi^*(t, s, \omega)W(t, s, \omega)\Phi(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &= \int_{\mathcal{M}} \langle W(s, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &\quad - \int_{\mathcal{M}} \left\langle \int_s^t \Phi^*(\tau, s, \omega)H(\tau, s, \omega)\Phi(\tau, s, \omega)g(\omega) d\tau, g(\omega) \right\rangle d\mathbf{P}(\omega) \\ &\leq \int_{\mathcal{M}} \langle W(s, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \leq -\gamma \mathbb{E}_2[g]^2, \end{aligned}$$

where we use (5.1) and the positivity of H . The last inequality gives

$$\mathbb{E}_2[\Phi(t, s, \cdot)g(\cdot)] \geq \gamma^{1/2} \alpha^{-1/2} \mathbb{E}_2[g].$$

On the other hand,

$$\begin{aligned} \beta \int_s^t \mathbb{E}_2[\Phi(\tau, s, \cdot)g(\cdot)]^2 d\tau &\leq \int_{\mathcal{M}} \left\langle \int_s^t \Phi^*(\tau, s, \omega)H(\tau, s, \omega)\Phi(\tau, s, \omega)g(\omega) d\tau, g(\omega) \right\rangle d\mathbf{P}(\omega) \\ &= \int_{\mathcal{M}} \langle W(s, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &\quad - \int_{\mathcal{M}} \langle \Phi^*(t, s, \omega)W(t, s, \omega)\Phi(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &\leq - \int_{\mathcal{M}} \langle \Phi^*(t, s, \omega)W(t, s, \omega)\Phi(t, s, \omega)g(\omega), g(\omega) \rangle d\mathbf{P}(\omega) \\ &\leq \alpha \mathbb{E}_2[\Phi(t, s, \cdot)g(\cdot)]^2, \end{aligned}$$

where we use (5.1) and the negativity of $W(t, s, \omega)$. It can be found that

$$\alpha \mathbb{E}_2[\Phi(t, s, \cdot)g(\cdot)]^2 \geq \beta \int_s^t \mathbb{E}_2[\Phi(\tau, s, \cdot)g(\cdot)]^2 d\tau \geq \beta \gamma \alpha^{-1}(t-s) \mathbb{E}_2[g]^2,$$

and so by Lemma 4.1 we get the desired result. $\square$

REFERENCES

- [1] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
- [2] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications (45), Cambridge University Press, 1992.
- [3] R. Datko, *Extending a theorem of A.M. Liapunov to Hilbert space*, J. Math. Anal. Appl. **32** (1970), 610–616.
- [4] R. Datko, *Uniform asymptotic stability of evolutionary processes in a Banach space*, SIAM J. Math. Anal. **3** (1972), 428–445.
- [5] P.V. Hai, *Discrete and continuous versions of Barbashin-type theorem of linear skew-evolution semiflows*, Appl. Anal. **90** (2011), 1897–1907.
- [6] P.V. Hai, *Two new approaches to Barbashin theorem*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **19** (2012), 773–798.
- [7] P.V. Hai, *Polynomial stability and polynomial instability for skew-evolution semi-flows*, Result. Math. **74** (2019), Article no. 175.
- [8] M. Megan, A.L. Sasu, B. Sasu, *On uniform exponential stability of linear skew-product semiflows in Banach spaces*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), 143–154.
- [9] M. Megan, A.L. Sasu, B. Sasu, *Exponential instability of linear skew-product semiflows in terms of Banach function spaces*, Results Math. **45** (2004), 309–318.
- [10] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin, Heidelberg, New York, 1991.
- [11] A. Pazy, *On the applicability of Lyapunov's theorem in Hilbert space*, SIAM J. Math. Anal. **3** (1972), 291–294.
- [12] C. Preda, P. Preda, *Lyapunov theorems for the asymptotic behavior of evolution families on the half-line*, Canad. Math. Bull. **54** (2011), 364–369.
- [13] S. Rolewicz, *On uniform N-equistability*, J. Math. Anal. Appl. **115** (1986), 434–441.
- [14] D. Stoica, M. Megan, *On nonuniform dichotomy for stochastic skew-evolution semiflows in Hilbert spaces*, Czechoslovak Math. J. **62** (2012), 879–887.
- [15] J.M.A.M. van Neerven, *Exponential stability of operators and operator semigroups*, J. Funct. Anal. **130** (1995), 293–309.
- [16] T. Yue, *Barbashin type characterizations for the uniform polynomial stability and instability of evolution families*, Georgian Math. J. **29** (2022), 953–966.
- [17] J. Zabczyk, *Remarks on the control of discrete-time distributed parameter systems*, SIAM J. Control **12** (1974), 721–735.

Pham Viet Hai
hai.phamviet@hust.edu.vn
 <https://orcid.org/0000-0001-9993-4070>

Faculty of Mathematics and Informatics
Hanoi University of Science and Technology
1 Dai Co Viet, Hanoi, Vietnam

Received: May 20, 2025.

Accepted: June 21, 2025.