

INTEGRAL REPRESENTATION OF SOLUTIONS TO DIRAC SYSTEMS

Łukasz Rzepnicki

Communicated by P.A. Cojuhari

Abstract. We introduce a novel integral form for a fundamental set of solutions to one-dimensional Dirac systems with an integrable potential and spectral parameter $\mu \in \mathbb{C}$. This method enables the construction of solutions that are analytic in μ within the half-plane $\operatorname{Im} \mu > -r$, $r \geq 0$ and $|\mu| \rightarrow \infty$. Consequently, we derive estimates for the solutions that remain valid not just within a horizontal strip but throughout the entire half-plane.

Keywords: Dirac system, integrable potential, integral equations, fundamental system of solutions.

Mathematics Subject Classification: 34L40, 45F05.

1. INTRODUCTION

In recent years, Dirac systems have been the subject of research from many different perspectives. Some classical results can be found in [2, 8, 15]. More recent developments concerning the operators associated with Dirac-type systems are presented in, for example, [1, 3, 19]. Other works address questions regarding the completeness of the system of eigenfunctions (see [13] and [14]) or the basis property (see [5] and [11]). The cited works represent only selected results, and a broader overview of research related to Dirac systems is provided in [17] and the references therein.

Such results can be obtained by understanding how the solutions of Dirac systems behave with respect to a spectral parameter $\mu \in \mathbb{C}$ within a specific region of the complex plane depending on the spectral problem under consideration. Most articles provide results for μ lying within a horizontal strip in the complex plane. These kinds of results are adequate for specific types of boundary conditions, such as regular or strictly regular ones. However, the first step in studying a given spectral problem is the construction of a fundamental system that is analytic with respect to the spectral parameter in the region under consideration.

The main goal of this work is to construct an analytic fundamental system in the half-plane $\operatorname{Im} \mu > -r$, $r \geq 0$. Our method also allows for an easy derivation of solution estimates valid in the half-plane. Once these results are obtained, extending

them to the entire complex plane, except for a bounded region around zero, becomes straightforward. Such results are very useful in the study of spectral problems in various function spaces.

Our approach relies on obtaining a suitable representation of the Dirac system using integral operators. The first part of this paper focuses on establishing properties of these integral operators and subsequently applying them to analyze fundamental solutions of Dirac systems.

For $Y = (y_1, y_2)^T$ we consider a one-dimensional Dirac system in the following form:

$$Y'(x) + J(x)Y(x) = A_\mu Y(x), \quad x \in [0, 1], \quad (1.1)$$

where

$$A_\mu = i\mu \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J(x) = \begin{bmatrix} 0 & \sigma_1(x) \\ \sigma_2(x) & 0 \end{bmatrix}. \quad (1.2)$$

Here $\mu \in \mathbb{C}$ is a spectral parameter, and complex-valued functions σ_j belong to Lebesgue space $L_p[0, 1]$ for $j = 1, 2$, where $1 \leq p < 2$. The solution Y is understood to be a function having entries from the space of absolutely continuous functions on $[0, 1]$ (i.e. from the Sobolev space $W_1^1[0, 1]$) satisfying (1.1) for a.e. $x \in [0, 1]$. In this paper, we focus on the construction of the fundamental system of solutions to the equation (1.1) given by the functions $w = (w_1, w_2)^T$ and $v = (v_1, v_2)^T$, which satisfy the conditions $w_1(0) = 1$, $w_2(1) = 0$ and $v_1(0) = 0$, $v_2(1) = 1$. The adopted notion of solutions and the assumptions on the functions σ_j enable the derivation of results for the boundary value problem in $L_p[0, 1]$ spaces at a later stage.

The system (1.1) can be expressed in the classical form

$$-iB_0 Y'(x) + Q(x)Y(x) = \lambda Y(x), \quad (1.3)$$

where $\lambda = -\mu$ and

$$B_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q(x) = \begin{bmatrix} 0 & i\sigma_1(x) \\ -i\sigma_2(x) & 0 \end{bmatrix}. \quad (1.4)$$

This is a form that has been studied in many works in the literature.

In [7] and [17], the solutions to (1.1) were analyzed. The first work provided precise asymptotic expressions for the fundamental solutions with a potential given by $\sigma_1, \sigma_2 \in L_2[0, 1]$, and demonstrated their use in studying the Sturm–Liouville problem with a singular potential. The second paper addressed the case where $\sigma_1, \sigma_2 \in L_p[0, 1]$ with $1 \leq p < 2$. In both studies, the spectral parameter was assumed to lie within a horizontal strip $|\operatorname{Im} \mu| \leq d$, $d > 0$, and the results were derived using a common integral representation of the fundamental matrix D of (1.1) satisfying $D(0) = I$. In [7], this representation takes the form

$$D(x) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu} [J(t) + Q(x, t)] dt, \quad (1.5)$$

where A_μ and J are given by (1.2). Q is a matrix function continuous on

$$\Delta := \{(x, t) \in \mathbb{R}^2 : 0 \leq t \leq x \leq 1\}. \quad (1.6)$$

Moreover, Q is a unique solution of the integral equation

$$Q(x, t) = \int_0^{x-t} J(t + \xi) J(\xi) d\xi - \int_0^{x-t} J(t + \xi) Q(t + \xi, \xi) d\xi. \quad (1.7)$$

In [17] due to assumptions on the potential J , the kernel Q was defined on a different Banach space than the space of continuous functions on Δ . It is essential to note that the kernel Q is independent of the spectral parameter μ , a property that facilitates the derivation of precise asymptotic formulas for the fundamental system of solutions. These formulas can subsequently be applied to the investigation of specific spectral problems. The conceptual framework for this method is derived from [15, Ch. 1, §2, Problem 5].

Other approaches to studying solutions to Dirac systems can be found in the literature. One such approach, based on Prüfer's substitution, is presented in [18] and [19]. A series of papers [9, 10, 12] provide results obtained using transformation operators.

In this paper, we use the method introduced in [6], which is presented in the next section. The main results of the article are Theorems 5.1, 5.2, and Corollaries 5.3 and 5.4, which provide, respectively, representations of the fundamental solutions of (1.1) as a power series (analytic with respect to μ) of the appropriate integral operators, along with their estimates. Results of this type will be employed in future work for the precise analysis of spectral problems related to Dirac systems, for which asymptotic formulas are required not only in the strip.

In what follows, we will not write the domain of functions in the notation of a given function space. For example, we will use the notation $\mathcal{C}$ for the space of continuous functions on $[0, 1]$ instead of $\mathcal{C}[0, 1]$ and similarly for L_p spaces.

2. INTEGRAL EQUATIONS

In this section, we present integral representations for the fundamental solutions of (1.1) for $\mu \in \mathbb{C}$ such that $\operatorname{Im} \mu > -r$, where $r \geq 0$. We will then examine the basic properties of the integral operators that appear in these representations.

If $Y = (y_1, y_2)^T$, the explicit form of the system (1.1) is

$$\begin{aligned} y_1' + \sigma_1 y_2 &= i\mu y_1, \\ y_2' + \sigma_2 y_1 &= -i\mu y_2. \end{aligned} \quad (2.1)$$

Next, noting that $\operatorname{Im} \mu > -r$, with $r \geq 0$, we apply general formulas for the solutions of ordinary differential equations to derive

$$\begin{aligned} y_1(x) &= C_1 e^{i\mu x} - \int_0^x e^{i\mu(x-t)} \sigma_1(t) y_2(t) dt, \\ y_2(x) &= C_2 e^{i\mu(1-x)} + \int_x^1 e^{-i\mu(x-t)} \sigma_2(t) y_1(t) dt, \end{aligned} \quad (2.2)$$

with certain constants C_1 and C_2 .

Let $w = (w_1, w_2)^T$ be the solution of (2.2) with specific constants $C_1 = 1$ and $C_2 = 0$. Then, it is obvious that $w_1(0) = 1$, $w_2(1) = 0$ and we can rewrite the system in the following way:

$$\begin{aligned} w_1(x) &= e^{i\mu x} - \int_0^x e^{i\mu(x-t)} \sigma_1(t) \int_t^1 e^{-i\mu(t-s)} \sigma_2(s) w_1(s) ds dt, \\ w_2(x) &= \int_x^1 e^{-i\mu(x-t)} \sigma_2(t) w_1(t) dt. \end{aligned} \quad (2.3)$$

By applying the substitution $w_1 = e^{i\mu \cdot} z_1$ and carrying out several straightforward transformations, we derive an integral equation for the function z_1 :

$$z_1 + K_1 z_1 = 1, \quad (2.4)$$

where an integral operator is given by

$$(K_1 z)(x) = \int_0^x \sigma_1(t) \int_t^1 e^{2i\mu(s-t)} \sigma_2(s) z(s) ds dt, \quad x \in [0, 1]. \quad (2.5)$$

In the case $C_1 = 0$ and $C_2 = 1$ we will denote the solution as $v = (v_1, v_2)^T$. Proceeding in the same way as for w , we get $v_1(0) = 0$, $v_2(1) = 1$ and

$$\begin{aligned} v_1(x) &= - \int_0^x e^{i\mu(x-t)} \sigma_1(t) v_2(t) dt, \\ v_2(x) &= e^{i\mu(1-x)} - \int_x^1 e^{-i\mu(x-t)} \sigma_2(t) \int_0^t e^{i\mu(t-s)} \sigma_1(s) v_2(s) ds dt. \end{aligned} \quad (2.6)$$

This time the appropriate substitution is $v_2 = e^{i\mu(1-\cdot)} z_2$, and it leads to the integral equation for z_2 :

$$z_2 + K_2 z_2 = 1, \quad (2.7)$$

where an integral operator is given by

$$(K_2 z)(x) = \int_x^1 \sigma_2(t) \int_0^t e^{2i\mu(t-s)} \sigma_1(s) z(s) ds dt, \quad x \in [0, 1]. \quad (2.8)$$

3. PRELIMINARY ESTIMATES

In the following, our main goal will be to examine the solvability of the equations (2.4) and (2.7) in the space $\mathcal{C}$ on the half-plane

$$\Sigma_r = \{\mu \in \mathbb{C} : \operatorname{Im} \mu > -r\}, \quad r \geq 0. \quad (3.1)$$

The results on the solvability of (1.1) can be obtained using classic methods as in [20, Chapter 1]. However, it is important to note that our integral equations are of the form $(I - K_j)z_j = 1$. Providing $\|K_j\|_{\mathcal{C}} < 1$ we can represent solutions z_j as a powers series in K_j . This is why it is necessary to derive estimates for the operators K_1 and K_2 with respect to a spectral parameter μ .

The estimates for these operators require an analysis of the expressions

$$\gamma_{0,1}(x, \mu) = \left| \int_0^x e^{2i\mu(x-t)} \sigma_1(t) dt \right|, \quad \gamma_{0,2}(x, \mu) = \left| \int_x^1 e^{2i\mu(t-x)} \sigma_2(t) dt \right|. \quad (3.2)$$

We will also need the following functions for $\sigma_j \in L_p[0, 1]$, $j = 1, 2$ and $1/p + 1/q = 1$, $q \in (1, \infty]$:

$$\begin{aligned} \gamma_1(\mu) &= \|\gamma_{0,1}(\cdot, \mu)\|_{L_q} = \left(\int_0^1 \left| \int_0^s e^{2i\mu(s-t)} \sigma_1(t) dt \right|^q ds \right)^{1/q}, \\ \gamma_2(\mu) &= \|\gamma_{0,2}(\cdot, \mu)\|_{L_q} = \left(\int_0^1 \left| \int_t^1 e^{2i\mu(s-t)} \sigma_2(s) ds \right|^q dt \right)^{1/q} \end{aligned} \quad (3.3)$$

and

$$v(r) = 1, \quad r < 0, \quad v(r) = e^{2r}, \quad r \geq 0. \quad (3.4)$$

If $p = 1$, then in (3.3) we set $\gamma_j(\mu)$ to be the supremum norm of $\gamma_{0,j}(\cdot, \mu)$ for $j = 1, 2$.

First, we would like to examine the behavior of gammas functions as $|\mu| \rightarrow \infty$. We define

$$\begin{aligned} \nu_{1,f}(x, \mu) &= \int_0^x e^{2i\mu(x-t)} f(t) dt, \\ \nu_{2,f}(x, \mu) &= \int_x^1 e^{2i\mu(t-x)} f(t) dt. \end{aligned}$$

In the next lemma, we will use the notion of $\mathcal{K}$ -functionals defined by

$$\mathcal{K}(f; \epsilon) = \inf_{g \in W_1^1} \{ \|f - g\|_{L_1} + \epsilon \|g\|_{W_1^1} \}, \quad \epsilon \in (0, 1], f \in L_1. \quad (3.5)$$

It is clear from the definition that

$$\lim_{\epsilon \rightarrow 0} \mathcal{K}(f; \epsilon) = 0.$$

More on $\mathcal{K}$ -functionals one can find in [4, Chapter VI] and [16].

Lemma 3.1. *For a given $f \in L_1$, and for $\mu \in \mathbb{C}$, $x \in [0, 1]$, the following estimates hold:*

$$\|\nu_{j,f}(\cdot, \mu)\|_C \leq (1 + v(r))\mathcal{K}(f; |\mu|^{-1}), \quad \mu \in \Sigma_r(1), \quad j = 1, 2, \quad (3.6)$$

where the function v is given by (3.4). What is more, we have

$$\|\nu_{j,f}(\cdot, \mu)\|_C \rightarrow 0, \quad \mu \in \Sigma_r, \quad |\mu| \rightarrow \infty. \quad (3.7)$$

Proof. Let us consider the case $j = 1$. It is straightforward that

$$|\nu_{1,f}(x, \mu)| \leq v(r)\|f\|_{L_1}, \quad f \in L_1.$$

On the other hand, for $g \in W_1^1$ and $\mu \neq 0$ integrating by parts we get

$$\begin{aligned} \nu_{1,g}(x, \mu) &= -\frac{1}{2i\mu} \int_0^x g(t) d e^{2i\mu(x-t)} \\ &= -\frac{1}{2i\mu} (g(x) - g(0)e^{2i\mu x}) + \frac{1}{2i\mu} \int_0^x e^{2i\mu(x-t)} g'(t) dt. \end{aligned}$$

Next, the integral representation of $f \in W_1^1$

$$f(x) = \int_0^x t f'(t) dt + \int_x^1 (t-1) f'(t) dt + \int_0^1 f(t) dt$$

implies that

$$\|f\|_C \leq \|f\|_{W_1^1} = \|f\|_{L_1} + \|f'\|_{L_1}, \quad f \in W_1^1. \quad (3.8)$$

This leads to the following conclusion for $\mu \in \Sigma_r$:

$$\|\nu_{1,g}(\cdot, \mu)\|_C \leq \frac{1 + v(r)}{2|\mu|} \|g\|_{W_1^1} + \frac{v(r)}{2|\mu|} \|g'(t)\|_{L_1} \leq \frac{1 + v(r)}{|\mu|} \|g\|_{W_1^1}.$$

Therefore, for $f \in L_1$ it follows that

$$\begin{aligned} \|\nu_{1,f}(\cdot, \mu)\|_C &\leq \|\nu_{1,f-g}(\cdot, \mu)\|_C + \|\nu_{1,g}(\cdot, \mu)\|_C \\ &\leq (1 + v(r))\{\|f - g\|_{L_1} + |\mu|^{-1} \|g\|_{W_1^1}\}. \end{aligned}$$

Summarizing the discussion above, for $\mu \in \Sigma_r(1)$, we obtain

$$\|\nu_{1,f}(\cdot, \mu)\|_C \leq \inf_{g \in W_1^1} \{\|\nu_{1,f-g}(\cdot, \mu)\|_C + \|\nu_{1,g}(\cdot, \mu)\|_C\} \leq (1 + v(r))\mathcal{K}(f; |\mu|^{-1}),$$

and this is exactly (3.6) for $j = 1$.

To prove (3.6) for $j = 2$, it is sufficient to observe that

$$\nu_{2,f}(x, \mu) = \nu_{1,\tilde{f}}(1 - x, \mu), \quad \tilde{f}(x) := f(1 - x), \quad x \in [0, 1],$$

and $\mathcal{K}(f; \cdot) = \mathcal{K}(\tilde{f}; \cdot)$, where $\mathcal{K}(f; \cdot)$ is given by (3.5). $\square$

A direct consequence of this lemma is the following result.

Corollary 3.2. *For $j = 1, 2$ and $\mu \in \Sigma_r$, $|\mu| \rightarrow \infty$, the following holds:*

$$\|\gamma_{0,j}(\cdot; \mu)\|_C \rightarrow 0 \quad \text{and} \quad \gamma_j(\mu) \rightarrow 0.$$

4. ESTIMATIONS OF INTEGRAL OPERATORS

In this section, we will focus on estimating integral operators. We begin with a few simple observations.

Lemma 4.1. *For $\sigma_j \in L_p$, $1 \leq p < q$ and $\mu \in \mathbb{C}$, the following estimates hold:*

$$\|K_1 e\|_C \leq \int_0^1 |\sigma_1(t)| \gamma_{0,2}(t, \mu) dt, \quad \|K_2 e\|_C \leq \int_0^1 |\sigma_2(t)| \gamma_{0,1}(t, \mu) dt, \quad (4.1)$$

$$\|K_1 e\|_C \leq \|\sigma_1\|_{L_p} \gamma_2(\mu), \quad \|K_2 e\|_C \leq \|\sigma_2\|_{L_p} \gamma_1(\mu). \quad (4.2)$$

Proof. The first inequality follows immediately from the definition of K_j . By Hölder inequality, for any $\sigma \in L_p$, we obtain

$$\int_0^1 |\sigma(t)| \gamma_{0,j}(t, \mu) dt \leq \|\sigma\|_{L_p} \gamma_j(\mu), \quad j = 1, 2, \quad (4.3)$$

and this gives (4.2). $\square$

In the following, we will use a specific decomposition of K_1 into two parts. Note that

$$(K_1 z)(x) = (K_{1,1} z)(x) + (K_{1,2} z)(x), \quad x \in [0, 1], z \in \mathcal{C}, \quad (4.4)$$

where

$$(K_{1,1} z)(x) = \int_0^x \sigma_2(s) \left\{ \int_0^s e^{2i\mu(s-t)} \sigma_1(t) dt \right\} z(s) ds, \quad (4.5)$$

$$(K_{1,2} z)(x) = \int_x^1 e^{2i\mu s} \sigma_2(s) z(s) ds \int_0^x e^{-2i\mu t} \sigma_1(t) dt. \quad (4.6)$$

We have gathered several simple inequalities in the following lemma.

Lemma 4.2. For $z \in \mathcal{C}$, $\mu \in \Sigma_r$, there hold

$$\|K_1 z\|_{\mathcal{C}} \leq v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_1} \|z\|_{\mathcal{C}}, \quad (4.7)$$

$$\|K_1 z\|_{\mathcal{C}} \leq v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_p} \|z\|_{L_q}, \quad (4.8)$$

where v is given by (3.4). Moreover, we have

$$\|K_{1,1} z\|_{\mathcal{C}} \leq \|\sigma_2\|_{L_p} \gamma_1(\mu) \|z\|_{\mathcal{C}}, \quad (4.9)$$

$$|(K_{1,2} z)(x)| \leq \|\sigma_2\|_{L_1} v(r) \gamma_{0,1}(x, \mu) \|z\|_{\mathcal{C}}, \quad x \in [0, 1], \quad (4.10)$$

and consequently

$$|(K_1 z)(x)| \leq \|\sigma_2\|_{L_p} (\gamma_1(\mu) + v(r) \gamma_{0,1}(x, \mu)) \|z\|_{\mathcal{C}}, \quad x \in [0, 1]. \quad (4.11)$$

Proof. For $\mu \in \Sigma_r$, we can easily deduce from (2.5) the inequality

$$\|K_1 z\|_{\mathcal{C}} \leq v(r) \int_0^1 |\sigma_1(t)| dt \int_0^1 |\sigma_2(s) z(s)| ds,$$

thus both estimates (4.7) and (4.8) follow. Next, using (4.3), we obtain

$$\begin{aligned} |(K_{1,1} z)(x)| &\leq \|z\|_{\mathcal{C}} \int_0^x |\sigma_2(s)| \left| \int_0^s e^{2i\mu(s-t)} \sigma_1(t) dt \right| ds \\ &= \|z\|_{\mathcal{C}} \int_0^x |\sigma_2(s)| \gamma_{0,1}(s, \mu) ds \leq \|z\|_{\mathcal{C}} \|\sigma_2\|_{L_p} \gamma_1(\mu), \end{aligned}$$

and similarly, we have

$$\begin{aligned} |(K_{1,2} z)(x)| &\leq \|z\|_{\mathcal{C}} \int_x^1 |\sigma_2(s)| \left| \int_0^x e^{2i\mu(s-t)} \sigma_1(t) dt \right| ds \\ &\leq \|z\|_{\mathcal{C}} v(r) \|\sigma_2\|_{L_1} \gamma_{0,1}(x, \mu). \end{aligned}$$

This completes the proof of (4.9) and (4.10). $\square$

Remark 4.3. The following decomposition holds for K_2 :

$$(K_2 z)(x) = (K_{2,1} z)(x) + (K_{2,2} z)(x), \quad x \in [0, 1], \quad z \in \mathcal{C}, \quad (4.12)$$

where

$$(K_{2,1} z)(x) = \int_x^1 \sigma_1(s) \left\{ \int_s^1 e^{2i\mu(t-s)} \sigma_2(t) dt \right\} z(s) ds, \quad (4.13)$$

$$(K_{2,2} z)(x) = \int_0^x e^{-2i\mu s} \sigma_1(s) z(s) ds \int_x^1 e^{2i\mu t} \sigma_2(t) dt. \quad (4.14)$$

Analogously as for K_1 , one can prove the following facts:

$$\|K_{2,1}z\|_C \leq \|\sigma_1\|_{L_p} \gamma_2(\mu) \|z\|_C, \quad (4.15)$$

$$|(K_{2,2}z)(x)| \leq \|\sigma_1\|_{L_1} v(r) \gamma_{0,2}(x, \mu) \|z\|_C, \quad (4.16)$$

and

$$|(K_2z)(x)| \leq \|\sigma_1\|_{L_p} (\gamma_1(\mu) + v(r) \gamma_{0,2}(x, \mu)) \|z\|_C, \quad x \in [0, 1], \quad (4.17)$$

where $z \in \mathcal{C}$, $\mu \in \Sigma_r$, $r \geq 0$.

The next result concerns the estimation of powers of the integral operator K_1 , which will be crucial in proving the solvability of the integral equations. For convenience, we introduce the notation:

$$\sigma(x) = \max\{|\sigma_1(x)|, |\sigma_2(x)|\} \in L_p,$$

and

$$e(x) = 1, \quad x \in [0, 1].$$

It is obvious that

$$\|\sigma_j\|_{L_1} \leq \|\sigma_j\|_{L_p} \leq \|\sigma\|_{L_p}$$

for $j = 1, 2$.

Lemma 4.4. *If $\mu \in \Sigma_r$ and $z \in \mathcal{C}$, then for $n = 0, 1, 2, \dots$ there holds*

$$\|K_1^{2n}z\|_C \leq a^n \gamma_1^n(\mu) \|z\|_C, \quad (4.18)$$

where $a := 2v^2(r) \|\sigma\|_{L_p}^3$ and v is given by (3.4).

Proof. We will proceed by induction. For $n = 0$, the estimate (4.18) is trivial. Next, suppose that (4.18) holds for some $n \in \mathbb{N}_0$. Then, using inequalities (4.7)–(4.10), we obtain

$$\begin{aligned} \|K_1^{2n+2}z\|_C &\leq \|K_1 K_{1,1} K_1^{2n}z\|_C + \|K_1 K_{1,2} K_1^{2n}z\|_C \\ &\leq v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_1} \|K_{1,1} K_1^{2n}z\|_C + v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_p} \|K_{1,2} K_1^{2n}z\|_{L_q} \\ &\leq v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_1} \|\sigma_2\|_{L_p} \gamma_1(\mu) \|K_1^{2n}z\|_C \\ &\quad + v(r) \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_p} v(r) \|\sigma_2\|_{L_1} \|\gamma_{0,1}(\cdot, \mu)\|_{L_q} \|K_1^{2n}z\|_C. \end{aligned}$$

Therefore, there holds

$$\|K_1^{2n+2}z\|_C \leq 2v^2(r) \|\sigma\|_{L_p}^3 \gamma_1(\mu) \|K_1^{2n}z\|_C \leq a^{n+1} \gamma_1^{n+1}(\mu) \|z\|_C,$$

and (4.18) is true for $n + 1$. $\square$

We can obtain analogous results for K_2 using a simple transformation. Note that if we take

$$\tilde{\sigma}_j(x) = \sigma_j(1 - x), \quad j = 1, 2, \quad \tilde{z}(x) = z(1 - x),$$

then

$$\begin{aligned}
 (K_2 z)(x) &= \int_x^1 \tilde{\sigma}_2(1-t) \left\{ \int_0^t \tilde{\sigma}_1(1-s) e^{2i\mu(t-s)} \tilde{z}(1-s) ds \right\} dt \\
 &= \int_x^1 \tilde{\sigma}_2(1-t) \left\{ \int_{1-t}^1 \tilde{\sigma}_1(\tau) e^{2i\mu(t+\tau-1)} \tilde{z}(\tau) d\tau \right\} dt \\
 &= \int_0^{1-x} \tilde{\sigma}_2(u) \left\{ \int_u^1 \tilde{\sigma}_1(\tau) e^{2i\mu(\tau-u)} \tilde{z}(\tau) d\tau \right\} du \\
 &= \int_0^{1-x} \tilde{\sigma}_2(t) \left\{ \int_t^1 \tilde{\sigma}_1(s) e^{2i\mu(s-t)} \tilde{z}(s) ds \right\} dt = (\tilde{K}_1 \tilde{z})(1-x).
 \end{aligned}$$

So, we can write

$$(K_{2,\sigma} z)(x) = (\tilde{K}_1 \tilde{z})(1-x), \quad x \in [0, 1], \quad (4.19)$$

where

$$(\tilde{K}_1 \tilde{z})(x) = \int_0^x \tilde{\sigma}_2(t) \int_t^1 e^{2i\mu(s-t)} \tilde{\sigma}_1(s) \tilde{z}(s) ds dt.$$

5. SOLVABILITY OF INTEGRAL EQUATIONS AND ESTIMATES

We are now ready to prove the main results of this paper. First, we will state the results regarding the solvability of equations (2.5) and (2.8) in $\mathcal{C}$. In the following, we will need the set

$$\Omega_r(R) = \Sigma_r \cap \{\mu \in \mathbb{C} : |\mu| > R\}. \quad (5.1)$$

Theorem 5.1. *Assume r is fixed and $R_r > 0$ be chosen large enough to satisfy*

$$\gamma_1(\mu) \leq 1/(2a) \quad \text{for } \mu \in \Omega_r(R_r). \quad (5.2)$$

Then, for $\mu \in \Omega_r(R_r)$ and any $f \in \mathcal{C}$, the equation

$$z(x) + (K_1 z)(x) = f(x),$$

has the unique solution $z \in \mathcal{C}$ given by

$$z = (1 + K_1)^{-1} f = \sum_{n=0}^{\infty} (-1)^n K_1^n f, \quad (5.3)$$

which is analytical with respect to μ and satisfies

$$\left\| z - \sum_{n=0}^3 (-1)^n K_1^n f \right\|_{\mathcal{C}} \leq 4a^2 (1 + v(r) \|\sigma\|_{L_p}^2) \gamma_1^2(\mu) \|f\|_{\mathcal{C}}. \quad (5.4)$$

Proof. The assertion (5.3) follows from (4.18) together with the condition (5.2). Note that $K_1 f$ is analytic with respect to μ for every continuous function f . Therefore, z is also analytical, as it is the uniform limit of the series in (5.3). Moreover, from (5.3), (4.7), (4.18) and (5.2), we obtain

$$\begin{aligned} \left\| z - \sum_{n=0}^3 (-1)^n K_1^n f \right\|_C &\leq \|K_1^4\|_C \sum_{n=0}^{\infty} \|K_1^n\|_C \|f\|_C \\ &\leq 2\|K_1^4\|_C (1 + \|K_1\|) \sum_{m=0}^{\infty} \|K_1^{2m}\|_C \|f\|_C \\ &\leq 2a^2 \gamma_1^2(\mu) (1 + v(r) \|\sigma\|_{L_p}^2) \sum_{m=0}^{\infty} a^m \gamma_1^m(\mu) \|f\|_C \\ &\leq 4a^2 (1 + v(r) \|\sigma\|_{L_p}^2) \gamma_1^2(\mu) \|f\|_C. \quad \square \end{aligned}$$

Combining together (4.19) and Theorem 5.1 we obtain the following assertions for K_2 .

Theorem 5.2. *Assume r is fixed and $R_r > 0$ be chosen large enough to satisfy*

$$\gamma_2(\mu) \leq 1/(2a) \quad \text{for } \mu \in \Omega_r(R_r). \quad (5.5)$$

Then, for $\mu \in \Omega_r(R_r)$ and any $f \in \mathcal{C}$, the equation

$$z(x) + (K_2 z)(x) = f(x),$$

has the unique solution $z \in \mathcal{C}$

$$z = (1 + K_2)^{-1} f = \sum_{n=0}^{\infty} (-1)^n K_2^n f, \quad (5.6)$$

which is analytical with respect to μ and satisfies

$$\left\| z - \sum_{n=0}^3 (-1)^n K_2^n f \right\|_C \leq 4a^2 (1 + v(r) \|\sigma\|_{L_p}^2) \gamma_2^2(\mu) \|f\|_C. \quad (5.7)$$

Note that the above results can be further refined in the case of equations (2.4) and (2.7) since their right-hand side is $e(x) = 1$, $x \in [0, 1]$. This simplifies the analysis of compositions of the operators K_j .

Corollary 5.3. *Let r be fixed and $R_r > 0$ satisfies (5.2). Then for $\mu \in \Omega_r(R_r)$ and the solution $z_1 \in \mathcal{C}$ of integral equation (2.4), we have the following estimates:*

$$\|z_1 - e + K_1 e - K_1^2 e\|_C \leq a \|\sigma\|_{L_p} (\gamma_2(\mu) + 4a(1 + v(r) \|\sigma\|_{L_p} \gamma_1(\mu)) \gamma_1(\mu)) \quad (5.8)$$

and

$$\begin{aligned} |z_1(x) - e(x) + (K_1 e)(x)| &\leq \|\sigma\|_{L_p}^2 (\gamma_1(\mu) + v(r) \gamma_{0,1}(x, \mu)) \gamma_2(\mu) \\ &\quad + a \|\sigma\|_{L_p} (\gamma_2(\mu) + 4a(1 + v(r) \|\sigma\|_{L_p} \gamma_1(\mu)) \gamma_1(\mu)). \end{aligned} \quad (5.9)$$

Proof. By (4.2) and (4.18), we have

$$\|K_1^3 e\|_{\mathcal{C}} \leq \|K_1^2\| \|K_1 e\| \leq a \|\sigma\|_{L_1} \gamma_1(\mu) \gamma_2(\mu),$$

hence (5.8) follows from (5.4). Next, using (4.2) and (4.11), we have for $x \in [0, 1]$

$$|(K_1^2 e)(x)| \leq \|\sigma_1\|_{L_1} \|\sigma_2\|_{L_p} (\gamma_1(\mu) + v(r) \gamma_{0,1}(x, \mu)) \gamma_2(\mu), \quad (5.10)$$

and then (5.9) follows from (5.8). $\square$

In the same way we can obtain results regarding the second operator.

Corollary 5.4. *Let r be fixed and $R_r > 0$ satisfies (5.5). Then for $\mu \in \Omega_r(R_r)$ and the solution $z_2 \in \mathcal{C}$ of integral equation (2.7) the following estimates are true:*

$$\|z_2 - e + K_1 e - K_2^2 e\|_{\mathcal{C}} \leq a \|\sigma\|_{L_p} (\gamma_1(\mu) + 4a(1 + v(r) \|\sigma\|_{L_p} \gamma_2(\mu)) \gamma_2(\mu) \quad (5.11)$$

and

$$\begin{aligned} |z_2(x) - e(x) + (K_2 e)(x)| &\leq \|\sigma\|_{L_p}^2 (\gamma_2(\mu) + v(r) \gamma_{0,2}(x, \mu)) \gamma_1(\mu) \\ &\quad + a \|\sigma\|_{L_p} (\gamma_1(\mu) + 4a(1 + v(r) \|\sigma\|_{L_p} \gamma_2(\mu)) \gamma_2(\mu)). \end{aligned} \quad (5.12)$$

We recall that solutions z_1 and z_2 of (2.4) and (2.7) are related to the fundamental system of solutions $w = (w_1, w_2)^T$ and $v = (v_1, v_2)^T$ to (2.2) through the transformations $w_1 = e^{i\mu \cdot} z_1$ and $v_2 = e^{i\mu(1-\cdot)} z_2$. Furthermore, w_2 and v_1 are given by (2.3) and (2.6), respectively. In this manner, we have derived results concerning the analyticity and estimates of the fundamental system of solutions to the Dirac system (1.1), which hold in any half-plane $\text{Im } \mu > -r$, $r \geq 0$ (for $|\mu|$ large enough).

Remark 5.5. In particular, the results are valid for $\text{Im } \mu \geq 0$ and outside some disk. It remains to show how to obtain results that hold in the entire complex plane, except for a disk around zero. Now, assuming that $\text{Im } \mu < 0$, we proceed by taking $\mu = -\lambda$ in (2.1). This way we get $\text{Im } \lambda > 0$ and

$$\begin{aligned} y_1' + \sigma_1 y_2 &= -i\lambda y_1, \\ y_2' + \sigma_2 y_1 &= i\lambda y_2. \end{aligned}$$

Therefore, it is sufficient to apply our results for $\text{Im } \mu > 0$ but replacing y_1 and y_2 and substituting $-\mu$ for μ and σ_2 for σ_1 in all the formulas.

Acknowledgements

The author would like to express sincere gratitude to the reviewer for their valuable time, insightful comments, and constructive suggestions which helped to improve the final version of the article.

REFERENCES

- [1] S. Albeverio, R. Hryniv, Y. Mykytyuk, *Inverse spectral problems for Dirac operators with summable potentials*, Russian Journal of Math. Physics **12** (2005), no. 4, 406–423.
- [2] G.D. Birkhoff, R.E. Langer, *The boundary problems and developments associated with a system of ordinary differential equations of the first order*, Proc. Amer. Acad. Arts Sci. **58** (1923), no. 4, 49–128.
- [3] J.C. Cuenin, P. Siegl, *Eigenvalues of one-dimensional non-self-adjoint Dirac operators and applications*, Letters in Mathematical Physics **108** (2018), 1757–1778.
- [4] R.A. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin Heidelberg, 1993.
- [5] P. Djakov, B. Mityagin, *Riesz bases consisting of root functions of 1D Dirac operators*, Proc. Amer. Math. Soc. **141** (2013), 1361–1375.
- [6] A.M. Gomilko, V.N. Pivovarchik, *Asymptotics of solutions of the Sturm–Liouville equation with respect to a parameter*, Ukrainian Math. J. **53** (2001), 866–885.
- [7] A.M. Gomilko, Ł. Rzepnicki, *On asymptotic behaviour of solutions of the Dirac system and applications to the Sturm–Liouville problem with a singular potential*, Journal of Spectral Theory **10** (2020), 747–786.
- [8] B.M. Levitan, I.S. Sargsyan, *Sturm–Liouville and Dirac Operators*, Kluwer Academic Publishers Group, Dordrecht, 1991.
- [9] A.A. Lunev, M.M. Malamud, *On the asymptotic expansion of the characteristic determinant for a 2×2 Dirac type system*, Journal of Mathematical Sciences **284** (2024), 795–823.
- [10] A.A. Lunyov, *Criterion of Bari basis property for 2×2 Dirac-type operators with strictly regular boundary conditions*, Math. Nachr. **296** (2023), 4125–4151.
- [11] A.A. Lunyov, M.M. Malamud, *On the Riesz basis property of root vectors system for 2×2 Dirac type operators*, J. Math. Anal. Appl. **441** (2016), 57–103.
- [12] A.A. Lunyov, M.M. Malamud, *Stability of spectral characteristics and Bari basis property of boundary value problems for 2×2 Dirac type systems*, Journal of Differential Equations **313** (2022), 633–742.
- [13] A.A. Lunyov, M.M. Malamud, *On the completeness property of root vector systems for 2×2 Dirac type operators with non-regular boundary conditions*, J. Math. Anal. Appl. **543** (2025).
- [14] A. Makin, *On the completeness of root function system of the Dirac operator with two-point boundary conditions*, Math. Nachr. **297** (2024), 2468–2487.
- [15] V.A. Marchenko, *Sturm–Liouville Operators and Their Applications*, Birkhauser, Basel, 1986.
- [16] G.V. Radzievskii, *Direct and inverse theorems on approximation by root functions of a regular boundary-value problem*, Sbornik: Mathematics **197** (2006), 1037–1083.
- [17] Ł. Rzepnicki, *Asymptotic behavior of solutions of the Dirac system with an integrable potential*, Integr. Equ. Oper. Theory **93** (2021), 1–24.

- [18] A.M. Savchuk, I.V. Sadovnichaya, *Asymptotic formulas for fundamental solutions of the Dirac system with complex-valued integrable potential*, Differential Equations **49** (2013), 545–556.
- [19] A.M. Savchuk, A.A. Shkalikov, *Dirac operator with complex-valued summable potential*, Math. Notes **96** (2014), 777–810.
- [20] A. Zettl, *Sturm–Liouville theory*, American Mathematical Society, Providence, 2005.

Łukasz Rzepnicki

keleb@mat.umk.pl

 <https://orcid.org/0000-0001-8532-6680>

Nicolaus Copernicus University in Toruń

Faculty of Mathematics and Computer Science

Chopina 12/18, 87-100 Toruń, Poland

Received: February 25, 2025.

Revised: April 29, 2025.

Accepted: May 2, 2025.