

NONLINEAR PARABOLIC EQUATION HAVING NONSTANDARD GROWTH CONDITION WITH RESPECT TO THE GRADIENT AND VARIABLE EXPONENT

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Abstract. We are concerned with the existence of solutions to a class of quasilinear parabolic equations having critical growth nonlinearity with respect to the gradient and variable exponent. Using Schaeffer's fixed point theorem combined with the sub- and supersolution method, we prove the existence results of a weak solutions to the considered problems.

Keywords: variable exponent, quasilinear equation, Schaeffer's fixed point, subsolution, supersolution, weak solution.

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1. INTRODUCTION

In the last decade, theoretical studies of partial differential equations have given birth to a new type of problem with nonstandard growth conditions. This new type of problem is often linked to the name “variable exponent” which means that the equation and their operator have a variable growth condition. Mathematical analysis of PDEs with variable exponent has undergone a great evolution in several fields of applied science, among which there are dynamics fluid, image processing [13, 15, 16, 29, 30], epidemiology models and their related predator-prey models [1, 6, 7]. The functional frameworks involving these type of problems are $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ called, respectively, Lebesgue and Sobolev space with variable exponent. For more details on these spaces, we refer the readers to [15, 20, 28].

The purpose of this work is to study the existence of a weak solution for a class of quasilinear parabolic equation with variable exponent modeled by

$$\begin{cases} \partial_t u - \operatorname{div}(A(t, x, \nabla u)) = f(t, x, u, \nabla u) & \text{in } Q_T := (0, T) \times \Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \\ u(t, x) = 0 & \text{on } \Sigma_T := (0, T) \times \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is an open bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $T > 0$, and the initial data u_0 is assumed to be a measurable function belonging in $L^2(\Omega)$. The operator $-\operatorname{div}(A(t, x, \nabla u))$ is of the type Leray–Lions with variable exponent $p(x)$. We assume that p is a continuous function on $\bar{\Omega}$ with $\inf_{x \in \bar{\Omega}} p(x) > 1$ and $A : Q_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying

- (H₁) $|A(t, x, \xi)| \leq H(t, x) + |\xi|^{p(x)-1}$,
- (H₂) $A(t, x, \xi)\xi \geq d|\xi|^{p(x)}$,
- (H₃) $\langle A(t, x, \xi) - A(t, x, \xi^*), \xi - \xi^* \rangle > 0$

for almost every (t, x) in Q_T and for every ξ, ξ^* in $\mathbb{R}^N$ ($\xi \neq \xi^*$), with $H \in L^{\frac{p(x)}{p(x)-1}}(Q_T)$ and $d > 0$. For the nonlinearity f , we assume that

- (H₄) $f : Q_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function,
- (H₅) $(s, r) \mapsto f(t, x, s, r)$ is locally Lipschitz continuous for a.e (t, x) in Q_T ,
- (H₆) $f(t, x, s, 0) = \min \{f(t, x, s, r), r \in \mathbb{R}^N\} = 0$.

Quasilinear partial differential equations has pulled the attention of several authors and great works have been published not only for initial data [3, 4, 17–19, 22–24, 27, 28] but also for stationary and periodic case (see for example the works [5, 10–12, 14]). To present the novelty and the originality of our work, we propose to recall some recent works which have been dealt with the particular cases of the problem (1.1). We start by the paper of Bendahmane *et al.* [8], where the authors studied (1.1) when u_0 belongs to $L^1(\Omega)$, f belongs to $L^1(Q_T)$ and does not depend on $(u, \nabla u)$. Based on the semigroup theory, they established well-posedness (existence and uniqueness) of a renormalized solution to (1.1). They proved that the obtained solution is also the entropy solution of the considered problem. Zhang and Zhou studied in [32] the existence-uniqueness of renormalized and entropy solution of the same equation (1.1). They used the semi-discretization time method to prove the well-posedness of an approximate weak solution to (1.1). Thereafter, they obtained the existence of a renormalized solution to (1.1) as a limit of an approximate problem. Based on the choice of the used test function, the authors showed the uniqueness of the obtained solution and they demonstrated the equivalence between the renormalized solution and the entropy solution to (1.1). The results of [8, 32] were generalized by Li and Gao in their paper [21], where they studied the existence of solutions to (1.1) with a particular sign assumption on the nonlinearity $f(u, \nabla u)$. Via the convergence of truncation, they obtained the existence of renormalized solution to the considered problem. In [22] Li *et al.* studied the equation (1.1) with a smooth initial condition and f depends only on ∇u . Under the De Giorgi iteration technique, the authors proved

the critical a priori L^∞ -estimates and thus established the existence of weak solutions to (1.1). Note that all these works examined the $p(x)$ -Laplacian operator which is a particular case of the considered operator in the equation (1.1). Therefore, the case of the Leray–Lions operator was discussed in the current literature. In particular, Ouaro and Ouedraogo proved in [24] the existence and uniqueness of the entropy solutions to (1.1) with L^1 -data. Their proof was based on the nonlinear semigroup theory and involved Lebesgue and Sobolev spaces with variable exponent. In view of the semilinear case of (1.1) (f depending only on u), Rădulescu *et al.* [19] proposed a qualitative analysis on the existence and uniqueness of a weak solution to (1.1). The authors assumed that $f(x, u)$ is a Carathéodory function with respect to x and locally Lipschitz with respect to u . Under a suitable assumption on the variable exponent, they established the existence and uniqueness of the weak solution to (1.1). The authors discussed also the global behavior of the obtained solutions, more precisely, the convergence to a stationary solution as $t \rightarrow \infty$.

L^2 -solutions for PDEs with variable exponent were also examined by several authors. In [2] Akagi and Matsuura proposed a mathematical analysis of parabolic $p(x)$ -Laplacian equation with L^2 data. Using the subdifferential calculus they proved the existence and uniqueness of L^2 -solution to the considered problem and they studied the large-time behavior of the obtained solution. Shangerganesh and Balachandran [30] considered the reaction-diffusion model with variable exponents and L^2 -data and without growth conditions on $(u, \nabla u)$. The authors studied the existence of weak solutions to the considered model when the nonlinearities do not depend on ∇u . Based on the standard Galerkin's method and the Gronwall lemma, the authors established the existence and uniqueness of a weak solution to the considered model. However, in contrast to the earlier mentioned works, here we present two existence results of a weak solution to the quasilinear parabolic equation (1.1). For the first one, we will assume that $f(u, \nabla u)$ is bounded in Q_T . Under the application of Schaeffer's fixed point theorem in a suitable Banach space, we prove the existence of a weak solution to (1.1). Concerning the second existence result, we will assume that $f(u, \nabla u)$ has a critical growth with respect to the gradient. By combining the truncation technics with the sub- and supersolution method, we establish the existence of a weak solution to (1.1).

We start initially with a recall in which we state some interesting results and properties of Lebesgue–Sobolev spaces with exponents variables. Thereafter, we prove in Section 3 the existence result of a weak solution to the proposed equation with bounded nonlinearity. This is done with the help of Schaeffer's fixed point theorem. In Section 4, we use the method of sub- and supersolution to consider an approximate problem of (1.1). The existence of a weak solution to the last one is ensured by the result of Section 3. After that, we give a suitable estimates on the approximate solutions and we pass to the limit in the approximate problem. Section 5 is devoted to some auxiliaries results. The first result concerns the existence and uniqueness result of a weak parabolic equation with L^2 data. The second one presents an interesting compactness result of a class of parabolic equations with variable exponent.

2. PRELIMINARIES RESULTS AND NOTATIONS

2.1. LEBESGUE-SOBOLEV SPACES WITH VARIABLE EXPONENT

We begin this section by a brief recall of Lebesgue and Sobolev spaces with variable exponent. Let $p : \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous function. We define

$$p^- = \inf_{x \in \bar{\Omega}} p(x) \quad \text{and} \quad p^+ = \sup_{x \in \bar{\Omega}} p(x).$$

Throughout this paper, we assume that

$$1 < p^- \leq p(x) \leq p^+ < \infty. \quad (2.1)$$

The variable exponent Lebesgue space is introduced as

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R}; u \text{ is measurable with } \rho_{p(x)}(u) < \infty \right\},$$

where $\rho_{p(x)}(\cdot)$ defines the following convex modular

$$\rho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} dx.$$

We equip the Lebesgue space $L^{p(x)}(\Omega)$ with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \alpha > 0 : \rho_{p(x)}\left(\frac{u}{\alpha}\right) \leq 1 \right\}.$$

By the hypothesis (2.1), the space $L^{p(x)}(\Omega)$ becomes a separable, uniformly convex Banach space. The dual space of $L^{p(x)}(\Omega)$ is introduced as $L^{p'(x)}(\Omega)$ with

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Let $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$. Then the following Hölder inequality

$$\int_{\Omega} |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)}$$

holds true. The following proposition gives useful and interesting properties of Lebesgue spaces with a variable exponent.

Proposition 2.1.

- (i) $\min \left\{ \|u\|_{L^{p(x)}(\Omega)}^{p^-}, \|u\|_{L^{p(x)}(\Omega)}^{p^+} \right\} \leq \rho_{p(x)}(u) \leq \max \left\{ \|u\|_{L^{p(x)}(\Omega)}^{p^-}, \|u\|_{L^{p(x)}(\Omega)}^{p^+} \right\}.$
- (ii) *If Ω is bounded, the inclusion result between $L^{p(x)}(\Omega)$ spaces still holds. Furthermore, if p_1, p_2 are two variables exponents such that $p_1(x) \leq p_2(x)$ almost everywhere in Ω , then we have the following continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*
- (iii) *Let $q \in C(\bar{\Omega})$ be such that $1 \leq q(x) < p^*(x)$ for all $x \in \bar{\Omega}$. Then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, where*

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \geq N. \end{cases}$$

To extend the variable exponent $p : \bar{\Omega} \rightarrow (1, \infty)$ to the general case $\bar{Q}_T = [0, T] \times \bar{\Omega}$, we set $p(t, x) := p(x)$ for all $(t, x) \in \bar{Q}_T$. Hence, the variable exponent Lebesgue space $L^{p(x)}(Q_T)$ is defined as follows:

$$L^{p(x)}(Q_T) = \left\{ u : Q_T \rightarrow \mathbb{R}; u \text{ is measurable with } \int_{Q_T} |u(t, x)|^{p(x)} dx dt < \infty \right\}.$$

Equipped with the norm

$$\|u\|_{L^{p(x)}(Q_T)} = \inf \left\{ \alpha > 0 : \int_{Q_T} \left| \frac{u(t, x)}{\alpha} \right|^{p(x)} dx dt \leq 1 \right\}$$

it is a separable, uniformly convex Banach space. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)^N \right\},$$

where its norm is given as follows:

$$\|u\|_{1,p(x)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Due to this norm, the space $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space. We assume that $p(x)$ satisfies the log-Hölder-continuity condition, i.e. there exists a constant C such that

$$|p(x_1) - p(x_2)| \leq \frac{C}{-\log |x_1 - x_2|} \quad \text{for all } x_1, x_2 \in \Omega \text{ with } |x_1 - x_2| < \frac{1}{2}. \quad (2.2)$$

Under the assumption (2.2) the space of smooth functions $\mathcal{C}_c^\infty(\Omega)$ is dense in the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. For the sake of convenience, we define

$W_0^{1,p(x)}(\Omega)$ as the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. For any $u \in W_0^{1,p(\cdot)}(\Omega)$, the $p(x)$ -Poincaré inequality

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)}$$

holds true, where the constant C depends only on p and Ω . Thus, we define the norm on $W_0^{1,p(x)}(\Omega)$ such that

$$\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

For more properties of Lebesgue and Sobolev spaces with variable exponent, we refer the reader to the book [28].

2.2. FUNCTIONAL FRAMEWORK AND DEFINITIONS

In this paragraph, we present the functional framework used in this work and we enunciate the notion of weak solution adapted to solve the problem (1.1).

For any $T \in (0, +\infty)$, we define the time space

$$L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) = \left\{ u \in L^{p(x)}(Q_T) : \int_0^T \|\nabla u\|_{p(x)}^{p^-} dt < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))} = \left(\int_0^T \|\nabla u\|_{p(x)}^{p^-} dt \right)^{\frac{1}{p^-}}.$$

Now, let us introduce the space $\mathcal{V}$ which is already considered in the studies of parabolic problems with variable exponent

$$\mathcal{V} = \left\{ v \in L^{p^-}(0, T; W_0^{1,p(x)}(\Omega)) : |\nabla v| \in L^{p(x)}(Q_T)^N \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{V}} = \|\nabla u\|_{L^{p(x)}(Q_T)}.$$

Due to the $p(x)$ -Poincaré inequality and the continuity of the embedding

$$L^{p(x)}(Q_T) \hookrightarrow L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))$$

the norm $\|\cdot\|_{\mathcal{V}}$ is equivalent to the following norm

$$\|v\|_{\mathcal{V}} = \|v\|_{L^{p^-}(0, T; W_0^{1,p(x)}(\Omega))} + \|\nabla v\|_{L^{p(x)}(Q_T)}.$$

The space $\mathcal{V}$ is a separable and reflexive Banach space and $\mathcal{V}^*$ denoted its dual space. Some interesting properties of the space $\mathcal{V}$ are stated in the following lemma.

Lemma 2.2 ([8]). *Let $\mathcal{V}$ be the space defined as above. Then:*

(i) *We have the following continuous dense embedding*

$$L^{p+}(0, T; W_0^{1,p(x)}(\Omega)) \hookrightarrow \mathcal{V} \hookrightarrow L^{p-}(0, T; W_0^{1,p(x)}(\Omega)). \quad (2.3)$$

In particular, since $\mathcal{C}_c^\infty(Q_T)$ is dense in $L^{p+}(0, T; W_0^{1,p(x)}(\Omega))$, it is dense in $\mathcal{V}$ and for the corresponding dual spaces we have

$$L^{(p-)'}(0, T; (W_0^{1,p(x)}(\Omega))^*) \hookrightarrow \mathcal{V}^* \hookrightarrow L^{(p+)'}(0, T; (W_0^{1,p(x)}(\Omega))^*). \quad (2.4)$$

(ii) *Moreover, the elements of $\mathcal{V}^*$ are represented as follow: For all $\zeta \in \mathcal{V}^*$, there exists $\xi = (\xi_1, \dots, \xi_N) \in (L^{p'(x)}(Q_T))^N$ such that $\zeta = \operatorname{div}(\xi)$ and*

$$\langle \zeta, \varphi \rangle_{\mathcal{V}^*, \mathcal{V}} = \int_{Q_T} \xi \nabla \varphi \, dx dt$$

for any $\varphi \in \mathcal{V}$. Furthermore, we have

$$\|\zeta\|_{\mathcal{V}^*} = \max\{\|\xi_i\|_{L^{p(x)}(Q_T)} : i = 1, \dots, N\}.$$

(iii) *For any $u \in \mathcal{V}$ the following relationship holds true*

$$\min \left\{ \|u\|_{\mathcal{V}}^{p^-}, \|u\|_{\mathcal{V}}^{p^+} \right\} \leq \int_{Q_T} |\nabla u|^{p(x)} \, dx dt \leq \max \left\{ \|u\|_{\mathcal{V}}^{p^-}, \|u\|_{\mathcal{V}}^{p^+} \right\}. \quad (2.5)$$

Definition 2.3. A measurable function $u : Q_T \rightarrow \mathbb{R}$ is said to be a weak solution to the problem (1.1) if it satisfies the following properties:

$$\begin{aligned} u &\in \mathcal{V} \cap L^\infty(Q_T), \quad \partial_t u \in \mathcal{V}^* + L^1(Q_T), \\ f(t, x, u, \nabla u) &\in L^1(Q_T), \quad u(0, x) = u_0(x) \text{ in } L^2(\Omega), \\ \int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} A(t, x, \nabla u) \nabla \varphi &= \int_{Q_T} f(t, x, u, \nabla u) \varphi \end{aligned}$$

for every test function $\varphi \in \mathcal{V} \cap L^\infty(Q_T)$.

Remark 2.4. According to the result of [8] we have the following embedding

$$\{u \in \mathcal{V} \cap L^\infty(Q_T); \partial_t u \in \mathcal{V}^* + L^1(Q_T)\} \hookrightarrow \mathcal{C}([0, T]; L^2(\Omega))$$

which gives that the initial condition makes sense in Definition 2.3.

Lemma 2.5 ([22]). *Assume that $\pi : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 piecewise function such that $\pi(0) = 0$ and $\pi' = 0$ outside a compact set. Let $\Pi(s) = \int_0^s \pi(\sigma) d\sigma$. If $u \in \mathcal{V}$ with $\partial_t u \in \mathcal{V}^* + L^1(Q_T)$, then*

$$\int_0^T \langle \partial_t u, \pi(u) \rangle \, dt = \langle \partial_t u, \pi(u) \rangle_{\mathcal{V}^* + L^1(Q_T), \mathcal{V} \cap L^\infty(Q_T)} = \int_{\Omega} \Pi(u(T)) \, dx - \int_{\Omega} \Pi(u(0)) \, dx.$$

Now we give some truncation functions which will be useful in this work. For every positive real number k , we set

$$T_k(s) = \min(k, \max(s, -k)) \quad \text{and} \quad S_k(r) = \int_0^r T_k(s) ds.$$

3. AN EXISTENCE RESULT WITH BOUNDED NONLINEARITY

The purpose of this section is to establish the existence of a weak solution to the problem (1.1) when the nonlinearity f is bounded almost everywhere. The following theorem is the main result of this section.

Theorem 3.1. *Under the hypotheses (H_1) – (H_6) we assume the existence of a nonnegative function $M \in L^\infty(Q_T)$ such that for a.e. (t, x) in Q_T ,*

$$|f(t, x, r, \xi)| \leq M(t, x) \quad \text{for all } (r, \xi) \in \mathbb{R} \times \mathbb{R}^N. \quad (3.1)$$

Then for every $u_0 \in L^2(\Omega)$, the problem (1.1) has a weak solution.

Proof. In order to prove the result of Theorem 3.1, we propose to apply Schaeffer fixed point method. We set $\mathcal{X} := [0, 1] \times \mathcal{V}$ and we consider the following mapping

$$\begin{aligned} \mathcal{H} : \mathcal{X} &\longrightarrow \mathcal{V}, \\ (\lambda, v) &\longmapsto u, \end{aligned}$$

where u is a weak solution of the following parabolic equation

$$\begin{cases} \partial_t u - \operatorname{div}((t, x, \nabla u)) = f(t, x, v, \lambda \nabla v) & \text{in } Q_T, \\ u(0, x) = \lambda u_0(x) & \text{in } \Omega, \\ u(t, x) = 0 & \text{on } \Sigma_T. \end{cases} \quad (3.2)$$

Due to the assumption (3.1), the function $f(t, x, v, \lambda \nabla v)$ belongs to $L^2(Q_T)$ and the initial condition λu_0 belongs to $L^2(Q_T)$. Moreover, for a fixed $(\lambda, v) \in \mathcal{X}$, we deduce from Lemma 5.1 the existence of a unique weak solution $u \in \mathcal{V}$ to the problem (3.2) in the sense that

$$\begin{aligned} \partial_t u &\in \mathcal{V}^* + L^2(Q_T), \quad u(0, x) = \lambda u_0(x) \text{ in } L^2(\Omega), \\ \int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} A(t, x, \nabla u) \nabla \varphi &= \int_{Q_T} f(t, x, v, \lambda \nabla v) \varphi \end{aligned} \quad (3.3)$$

for every test function $\varphi \in \mathcal{V} \cap L^2(Q_T)$. As a consequence, the mapping $\mathcal{H}$ is well defined. Furthermore, from the assumption (H_6) and (3.3), it is easy to verify that for all $v \in \mathcal{V}$, we have $\mathcal{H}(0, v) = 0$. We set

$$\mathcal{U} = \left\{ u \in \mathcal{V} : u = \mathcal{H}(\lambda, u) \text{ for some } \lambda \in [0, 1] \right\}.$$

To apply Schaeffer's fixed point theorem, we proceed by three steps.

Step 1. The mapping $\mathcal{H}$ is continuous. Let (λ_n, v_n) be a sequence in $\mathcal{X}$ such that

$$(\lambda_n, v_n) \rightarrow (\lambda, v) \text{ strongly in } \mathcal{X}.$$

Let us define $u_n = \mathcal{H}(\lambda_n, v_n)$, which means that u_n satisfies the following weak formulation

$$\begin{aligned} \partial_t u_n \in \mathcal{V}^* + L^2(Q_T), \quad u_n(0, x) = \lambda_n u_0(x) \text{ in } L^2(\Omega), \\ \int_0^T \langle \partial_t u_n, \varphi \rangle + \int_{Q_T} A(t, x, \nabla u_n) \nabla \varphi = \int_{Q_T} f(t, x, v_n, \lambda_n \nabla v_n) \varphi \end{aligned} \quad (3.4)$$

for all $\varphi \in \mathcal{V} \cap L^2(Q_T)$. To prove the continuity of $\mathcal{H}$ it suffices to prove that (u_n) converges strongly to u in $\mathcal{V}$. According to the result of Lemma 5.1, one obtains

$$\|u_n\|_{\mathcal{V}} \leq C(\Omega, T) (\|\lambda_n u_0\|_{L^2(\Omega)} + \|f(t, x, v_n, \lambda_n \nabla v_n)\|_{L^2(Q_T)})$$

and

$$\|\partial_t u_n\|_{\mathcal{V}^* + L^2(Q_T)} \leq C(\Omega, T) (\|H\|_{p'(x)} + \|\lambda_n u_0\|_{L^2(\Omega)} + \|f(t, x, v_n, \lambda_n \nabla v_n)\|_{L^2(Q_T)}).$$

By the assumption (3.1), it follows that (u_n) is bounded in $\mathcal{V}$ and $(\partial_t u_n)$ is bounded in $\mathcal{V}^* + L^2(Q_T)$. On the other hand, due to the compactness result of Lemma 5.2, there exists a subsequence of (u_n) , still denoted by (u_n) for simplicity, such that

$$\begin{aligned} u_n &\rightarrow u \text{ strongly in } L^{p^-}(Q_T) \text{ and a.e. in } Q_T, \\ \nabla u_n &\rightarrow \nabla u \text{ a.e. in } Q_T. \end{aligned} \quad (3.5)$$

Therefore,

$$A(t, x, \nabla u_n) \rightharpoonup A(t, x, \nabla u) \text{ weakly in } L^{p'(x)}(Q_T).$$

From the strong convergence of (λ_n, v_n) in $\mathcal{X}$, it follows that

$$f(t, x, v_n, \lambda_n \nabla v_n) \rightarrow f(t, x, v, \lambda \nabla v) \text{ a.e. in } Q_T.$$

Using hypotheses (3.1) and the Lebesgue convergence theorem, we deduce that

$$f(t, x, v_n, \lambda_n \nabla v_n) \rightarrow f(t, x, v, \lambda \nabla v) \text{ strongly in } L^{(p^-)'}(Q_T) \quad (3.6)$$

We subtract the equation (3.4) for different indexes n and m , one gets

$$\begin{aligned} \int_0^T \langle \partial_t (u_n - u_m), \varphi \rangle + \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u_m)) \nabla \varphi \\ = \int_{Q_T} (f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v_m, \lambda_m \nabla v_m)) \varphi. \end{aligned}$$

Setting $\varphi = (u_n - u_m)$, one obtains

$$\begin{aligned} & \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u_m))(\nabla u_n - \nabla u_m) \\ & \leq \frac{1}{2} \int_{\Omega} |(\lambda_n u_0 - \lambda_m u_0)|^2 + \int_{Q_T} (f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v_m, \lambda_m \nabla v_m))(u_n - u_m) \end{aligned} \quad (3.7)$$

Using Hölder's inequality on the right-hand side of (3.7), we get

$$\begin{aligned} & \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u_m))(\nabla u_n - \nabla u_m) \leq \frac{|\lambda_n - \lambda_m|^2}{2} \int_{\Omega} |u_0|^2 \\ & + \|f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v_m, \lambda_m \nabla v_m)\|_{L^{(p-)'}(Q_T)} \|u_n - u_m\|_{L^{p-}(Q_T)}. \end{aligned} \quad (3.8)$$

By employing the almost everywhere convergence of (∇u_m) in Q_T , the assumption (H_3) and (3.6), we may employ Fatou's Lemma to pass to the limit in (3.8) as $m \rightarrow \infty$, one has

$$\begin{aligned} & \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u))(\nabla u_n - \nabla u) \\ & \leq \frac{|\lambda_n - \lambda|^2}{2} \int_{\Omega} |u_0|^2 \\ & + \|f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v, \lambda \nabla v)\|_{L^{(p-)'}(Q_T)} \|u_n - u\|_{L^{p-}(Q_T)}. \end{aligned} \quad (3.9)$$

From (3.5) and (3.6) it follows that

$$\lim_{n \rightarrow \infty} \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u))(\nabla u_n - \nabla u) \leq 0.$$

In view of the result [9], we deduce that (u_n) converges strongly to u in $\mathcal{V}$. Passing to the limit in (3.4), one gets

$$\begin{aligned} & \partial_t u \in \mathcal{V}^* + L^2(Q_T), \quad u(0, x) = \lambda u_0(x) \text{ in } L^2(\Omega) \\ & \int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} A(t, x, \nabla u) \nabla \varphi = \int_{Q_T} f(t, x, v, \lambda \nabla v) \varphi \end{aligned} \quad (3.10)$$

for all $\varphi \in \mathcal{V} \cap L^2(Q_T)$. Using the uniqueness of the weak solution of (3.10), we deduce that $\mathcal{H}(\lambda, v) = u$, which proves the continuity of $\mathcal{H}$.

Step 2. The mapping $\mathcal{H}$ is compact. We consider (λ_n, v_n) a bounded sequence in $\mathcal{X}$, we aim to prove that $u_n = \mathcal{H}(\lambda_n, v_n)$ is relatively compact in $\mathcal{V}$. Let us observe that

$$\begin{aligned} & \lambda_n \rightarrow \lambda^*, \\ & v_n \rightharpoonup v \text{ weakly in } \mathcal{V}. \end{aligned}$$

In this step, the difficulties come back in the absence of the almost everywhere convergence of (∇v_n) in Q_T , but we can overcome these difficulties by employing the assumption (3.1). By following the same reasoning of the first step, one gets:

- (a) u_n is bounded in $\mathcal{V}$,
- (b) $\partial_t u_n$ is bounded in $\mathcal{V}^* + L^2(Q_T)$,
- (c) $(f(t, x, v_n, \lambda_n \nabla v_n))_n$ is bounded in $L^2(Q_T)$.

Thanks to the compactness result of Lemma 5.2, there exist a subsequence still denoted by u_n for simplicity such that for

$$\begin{aligned} u_n &\rightarrow u \text{ strongly in } L^{p^-}(Q_T) \text{ and a.e. in } Q_T, \\ \nabla u_n &\rightarrow \nabla u \text{ and a.e. in } Q_T. \end{aligned}$$

Furthermore, we have

$$A(t, x, \nabla u_n) \rightharpoonup A(t, x, \nabla u) \text{ weakly in } L^{p'(x)}(Q_T).$$

We shall prove that (u_n) converges strongly in $\mathcal{V}$. We follow the same reasoning of the first step, for different index m and n , one has

$$\begin{aligned} &\int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u_m))(\nabla u_n - \nabla u_m) \\ &\leq \frac{|\lambda_n - \lambda_m|^2}{2} \int_{\Omega} |u_0|^2 \\ &\quad + \int_{Q_T} (f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v_m, \lambda_m \nabla v_m))(u_n - u_m). \end{aligned} \tag{3.11}$$

To deal with the right-hand side of (3.11), we apply the assumption (3.1) and Hölder's inequality, and we get

$$\begin{aligned} \int_{Q_T} (f(t, x, v_n, \lambda_n \nabla v_n) - f(t, x, v_m, \lambda_m \nabla v_m))(u_n - u_m) &\leq 2\|M\|_{L^\infty(Q_T)} \int_{Q_T} |u_n - u_m| \\ &\leq C\|u_n - u_m\|_{L^{p^-}(Q_T)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u_m))(\nabla u_n - \nabla u_m) &\leq \frac{|\lambda_n - \lambda_m|^2}{2} \int_{\Omega} |u_0|^2 \\ &\quad + C\|u_n - u_m\|_{L^{p^-}(Q_T)}. \end{aligned} \tag{3.12}$$

In view of the almost everywhere convergence of (∇u_m) and thanks to the assumption (H_3) , we can apply Fatou's Lemma to pass to the limit in (3.12) as $m \rightarrow \infty$. As a consequence, we obtain

$$\begin{aligned} \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u))(\nabla u_n - \nabla u) &\leq \frac{|\lambda_n - \lambda^*|^2}{2} \int_{\Omega} |u_0|^2 \\ &+ C \|u_n - u\|_{L^{p^-}(Q_T)}. \end{aligned} \quad (3.13)$$

Using the strong convergence of (u_n) in $L^{p^-}(Q_T)$, we deduce that

$$\lim_{n \rightarrow \infty} \int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla u))(\nabla u_n - \nabla u) \leq 0.$$

With the help of the result of [9], we conclude that (u_n) converges strongly to u in $\mathcal{V}$ which implies the compactness of the mapping $\mathcal{H}$.

Step 3. The set $\mathcal{U}$ is bounded in $\mathcal{V}$. Let $u \in \mathcal{V}$ such that $u = \mathcal{H}(\lambda, u)$ for some $\lambda \in [0, 1]$, we aim to prove that u is bounded in $\mathcal{V}$ independently of λ . By taking $\varphi = u$ as a test function in (3.3), we have

$$\frac{1}{2} \int_{\Omega} u^2(T) + \int_{Q_T} A(t, x, \nabla u) \nabla u = \frac{\lambda^2}{2} \int_{\Omega} u_0^2 + \int_{Q_T} f(t, x, u, \lambda \nabla u) u.$$

Thanks to the coercivity assumption (H_2) and by using (3.1), we get

$$d \int_{Q_T} |\nabla u|^{p(x)} \leq \int_{\Omega} u_0^2 + \int_{Q_T} |Mu|.$$

Hölder's inequality leads to

$$d \int_{Q_T} |\nabla u|^{p(x)} \leq \|u_0\|_{L^2(\Omega)} + C \|M\|_{L^\infty(Q_T)} \|u\|_{L^{p^-}(Q_T)}.$$

Applying the result of (2.3) and (2.5), one has

$$\min \left\{ \|u\|_{\mathcal{V}}^{p^-}, \|u\|_{\mathcal{V}}^{p^+} \right\} \leq C (\|u_0\|_{L^2(\Omega)} + \|M\|_{L^\infty(Q_T)} \|u\|_{\mathcal{V}}).$$

Using Young's inequality, one obtains

$$\min \left\{ \left(\frac{p^- - 1}{p^-} \right) \|u\|_{\mathcal{V}}^{p^-}, \left(\frac{p^+ - 1}{p^+} \right) \|u\|_{\mathcal{V}}^{p^+} \right\} \leq C,$$

where C is a constant depending only on $T, \Omega, p^-, p^+, d, \|u_0\|_{L^2(\Omega)}$ and $\|M\|_{L^\infty(Q_T)}$. As a consequence, $\mathcal{U}$ is bounded in $\mathcal{V}$. Then a direct application of Schaeffer's fixed point theorem (see e.g [25]) permits us to deduce the existence of a weak solution to the problem (1.1). $\square$

4. AN EXISTENCE RESULT WITH NONSTANDARD GROWTH NONLINEARITY

In this section, we are concerned by the existence result of a weak solution to (1.1) in the case when the nonlinearity f is nonnegative and has a critical growth with respect to the gradient namely

$$|f(t, x, r, \xi)| \leq c(|r|) \left[G(t, x) + |\xi|^{p(x)} \right], \quad (4.1)$$

where $c : [0, +\infty) \rightarrow [0, +\infty)$ is a non-decreasing function and G is a nonnegative function belonging to $L^1(Q_T)$.

Under the assumption that an order couple of sub- and supersolution existent, we prove the existence of a weak solution to (1.1), which is a SOLA solution (a solution obtained as a limit of approximation). First of all, let us define the notion of weak sub- and supersolution to (1.1).

Definition 4.1.

- (i) A weak subsolution of problem (1.1) is a measurable function $\underline{u} : Q_T \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} \underline{u} &\in \mathcal{V} \cap L^\infty(Q_T), \quad \partial_t \underline{u} \in \mathcal{V}^* + L^1(Q_T), \\ f(t, x, \underline{u}, \nabla \underline{u}) &\in L^1(Q_T), \quad \underline{u}(0, x) \leq u_0(x) \text{ in } L^2(\Omega), \\ \int_0^T \langle \partial_t \underline{u}, \varphi \rangle + \int_{Q_T} A(t, x, \nabla \underline{u}) \nabla \varphi &\leq \int_{Q_T} f(t, x, \underline{u}, \nabla \underline{u}) \varphi \end{aligned} \quad (4.2)$$

for every nonnegative test function $\varphi \in \mathcal{V} \cap L^\infty(Q_T)$.

- (ii) A weak supersolution of problem (1.1) is a measurable function $\bar{u} : Q_T \rightarrow \mathbb{R}$ satisfying (4.2) with $\leq$ is replaced by $\geq$.

In the following theorem, we state the main result of this section.

Theorem 4.2. *Assume that (H_1) – (H_6) and the nonlinearity f satisfies the growth assumption (4.1). Moreover, we assume the existence of $(\underline{u}, \bar{u})$ sub- and super solution such as $\underline{u} \leq \bar{u}$. Then, for any $u_0 \in L^\infty(\Omega)$ such that $\underline{u}(0) \leq u_0 \leq \bar{u}(0)$, the system (1.1) has a weak solution u such that $\underline{u} \leq u \leq \bar{u}$ a.e. in Q_T .*

To establish the result of Theorem 4.2, we will truncate the nonlinearity $f(t, x, u, \nabla u)$ to become bounded, thereafter we consider an approximate problem of (1.1). The existence of a weak solution of the last one will be proved by applying the result of Section 3. Thereafter, to pass to the limit in the approximate problem, we will provide necessary a priori estimates on the approached solution.

4.1. APPROXIMATE PROBLEM

Let $\underline{u}$ and $\bar{u}$ be, respectively, the sub- and supersolution of the problem (1.1). We introduce for all $u \in \mathcal{V}$ the following truncation function

$$\mathcal{T}(u) = u - (u - \bar{u})^+ + (\underline{u} - u)^+.$$

For any $n \geq 0$, we define the truncation function $\psi_n \in \mathcal{C}_c^\infty(\mathbb{R})$ such as $0 \leq \psi_n \leq 1$ and

$$\psi_n(s) = \begin{cases} 1 & \text{if } |s| \leq n, \\ 0 & \text{if } |s| \geq n+1. \end{cases}$$

For almost all $(t, x) \in Q_T$ and for all $(r, \xi) \in \mathbb{R} \times \mathbb{R}^N$, we approximate f by

$$f_n(t, x, u, \nabla u) = \psi_n(|u| + \|\nabla u\|) f(t, x, \mathcal{T}(u), \nabla \mathcal{T}(u)). \quad (4.3)$$

It is easy to verify that these functions f_n satisfy the properties (H_4) – (H_6) . Moreover, from (H_5) and (4.3) we deduce that $|f_n| \leq M_n$, where M_n is a constant depending only on n . Now, we can define the approximate problem of (1.1) as follows:

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t, x, \nabla u_n)) = f_n(t, x, u_n, \nabla u_n) & \text{in } Q_T, \\ u_n(0, x) = u_0(x) & \text{in } \Omega, \\ u_n(t, x) = 0 & \text{on } \Sigma_T. \end{cases} \quad (4.4)$$

From Theorem 3.1 we obtain the existence of u_n a weak solution to the approximate problem (4.4). In the following lemma we will prove that u_n is between $\underline{u}$ and $\bar{u}$, respectively, the sub- and supersolution of (1.1). This estimate leads to the fact that u_n belongs to $L^\infty(Q_T)$.

Lemma 4.3. *Let u_n be the weak solution of the approximate problem (4.4), then*

$$\underline{u} \leq u_n \leq \bar{u} \text{ a.e. in } Q_T. \quad (4.5)$$

Proof. Let us prove that $u_n \leq \bar{u}$ a.e. in Q_T . It is clear that $(u_n - \bar{u})^+ \in \mathcal{V} \cap L^\infty(\Omega)$. Then we can choose $\varphi = (u_n - \bar{u})^+$ as a test function in the weak formulation of (4.4). We obtain

$$\int_0^T \langle \partial_t u_n, (u_n - \bar{u})^+ \rangle + \int_{Q_T} A(\cdot, \nabla u_n) \nabla (u_n - \bar{u})^+ = \int_{Q_T} f_n(\cdot, u_n, \nabla u_n) (u_n - \bar{u})^+. \quad (4.6)$$

Since $\bar{u}$ is a supersolution of the problem (1.1), then we have

$$\int_0^T \langle \partial_t \bar{u}, (u_n - \bar{u})^+ \rangle + \int_{Q_T} A(\cdot, \nabla \bar{u}) \nabla (u_n - \bar{u})^+ \geq \int_{Q_T} f(\cdot, \bar{u}, \nabla \bar{u}) (u_n - \bar{u})^+. \quad (4.7)$$

By subtracting (4.7) from (4.6), we get

$$\begin{aligned} & \int_0^T \langle \partial_t(u_n - \bar{u}), (u_n - \bar{u})^+ \rangle + \int_{Q_T} (A(\cdot, \nabla u_n) - A(\cdot, \nabla \bar{u})) \nabla(u_n - \bar{u})^+ \\ & \leq \int_{Q_T} (f_n(\cdot, u_n, \nabla u_n) - f(\cdot, \bar{u}, \nabla \bar{u})) (u_n - \bar{u})^+. \end{aligned} \quad (4.8)$$

To deal with the first integral of (4.8) one may use Lemma 2.5. It turns out that

$$\int_0^T \langle \partial_t(u_n - \bar{u}), (u_n - \bar{u})^+ \rangle = \int_{\Omega} \Pi((u_n - \bar{u})(T)) dx - \int_{\Omega} \Pi((u_n - \bar{u})(0)) dx,$$

where in this case

$$\Pi(y) = \int_0^y s^+ ds.$$

Since $\bar{u}$ is a weak supersolution of (1.1), one may deduce that $(u_n - \bar{u})(0) \leq 0$. Then $\Pi((u_n - \bar{u})(0)) \leq 0$. Therefore, one gets

$$\int_0^T \langle \partial_t(u_n - \bar{u}), (u_n - \bar{u})^+ \rangle \geq 0.$$

For the right-hand side of (4.8), one may utilize (4.3) to obtain

$$\begin{aligned} & \int_{Q_T} (f_n(\cdot, u_n, \nabla u_n) - f(\cdot, \bar{u}, \nabla \bar{u})) (u_n - \bar{u})^+ \\ & \leq \int_{Q_T} (f(t, x, \mathcal{T}(u_n), \nabla \mathcal{T}(u_n)) - f(t, x, \bar{u}, \nabla \bar{u})) (u_n - \bar{u})^+ \\ & \leq \int_{\{u_n \geq \bar{u}\}} (f(t, x, \bar{u}, \nabla \bar{u}) - f(t, x, \bar{u}, \nabla \bar{u})) (u_n - \bar{u}) = 0. \end{aligned}$$

Therefore, we have

$$\int_{Q_T} (A(t, x, \nabla u_n) - A(t, x, \nabla \bar{u})) \nabla(u_n - \bar{u})^+ \leq 0$$

which implies that

$$\int_{\{u_n \geq \bar{u}\}} (A(t, x, \nabla u_n) - A(t, x, \nabla \bar{u})) \nabla(u_n - \bar{u}) \leq 0.$$

Using the property (H_3) , one gets $\nabla(u_n - \bar{u}) = 0$ a.e. in the set $\{(t, x) \in Q_T, u_n \geq \bar{u}\}$. Consequently, $u_n = \bar{u}$ a.e. in the set $\{(t, x) \in Q_T, u_n \geq \bar{u}\}$ which implies that $u_n \leq \bar{u}$ a.e. in Q_T .

By using similar reasoning of the first proof, we can obtain $\underline{u} \leq u_n$ a.e. in Q_T . $\square$

Remark 4.4. Note that the estimate (4.5) plays a crucial role in our work since it is helpful in several steps of the proof of a priori estimates. Moreover, from (4.5) one may deduce that

$$\|u_n\|_\infty \leq \|\underline{u}\|_\infty + \|\bar{u}\|_\infty := \Lambda$$

which implies that (u_n) is bounded in $L^\infty(Q_T)$.

4.2. A PRIORI ESTIMATES

First of all, we give a technical lemma which is frequently used in what follows.

Lemma 4.5. *Let $\theta(s) = se^{\eta s^2}$, $s \in \mathbb{R}$, and let $\Theta(s) = \int_0^s \theta(\tau) d\tau$. Then*

$$\theta(0) = 0, \quad \Theta(s) \geq 0, \quad \theta'(s) > 0.$$

When $\eta \geq \frac{b^2}{4a^2}$ is fixed, the following relationships hold true

$$a\theta'(s) - b|\theta(s)| \geq \frac{a}{2}, \quad s \in \mathbb{R}. \quad (4.9)$$

Lemma 4.6. *Let u_n be the sequence defined as above. Then there exists a constant C independent of n such that*

$$\begin{aligned} \|u_n\|_{\mathcal{V}} &\leq C, \\ \|f_n(t, x, u_n, \nabla u_n)\|_{L^1(Q_T)} &\leq C, \\ \|(\partial_t u_n)\|_{\mathcal{V}^* + L^1(Q_T)} &\leq C. \end{aligned}$$

Proof. Using the estimate (4.5), one may deduce that $\theta(u_n) \in \mathcal{V} \cap L^\infty(Q_T)$, then by taking $\theta(u_n)$ as a test function in the weak formulation of (4.4), we obtain

$$\int_0^T \langle \partial_t u_n, \theta(u_n) \rangle + \int_{Q_T} A(t, x, \nabla u_n) \nabla(u_n) \theta'(u_n) = \int_{Q_T} f_n(t, x, u_n, \nabla u_n) \theta(u_n). \quad (4.10)$$

For the first integral, we have

$$\int_0^T \langle \partial_t u_n, \theta(u_n) \rangle = \int_\Omega [\Theta(u_n(T)) - \Theta(u_0)].$$

Then, from (H_2) and (4.5) the inequality (4.10) becomes

$$\begin{aligned}
& \int_{\Omega} \Theta(u_n(T)) + d \int_{Q_T} |\nabla u_n|^{p(x)} \theta'(u_n) \\
& \leq \int_{\Omega} \Theta(u_0) + \int_{Q_T} |f_n(t, x, u_n, \nabla u_n) \theta(u_n)| \\
& \leq \int_{\Omega} \Theta(u_0) + \int_{Q_T} c(|u_n|) \left(G(t, x) + |\nabla u_n|^{p(x)} \right) |\theta(u_n)| \\
& \leq \int_{\Omega} \Theta(u_0) + c(\Lambda) \int_{Q_T} \left(G(t, x) + |\nabla u_n|^{p(x)} \right) |\theta(u_n)|.
\end{aligned}$$

We rewrite the above inequality as

$$\begin{aligned}
& \int_{\Omega} \Theta(u_n(T)) + \int_{Q_T} \left(d \theta'(u_n) - c(\Lambda) |\theta(u_n)| \right) |\nabla u_n|^{p(x)} \\
& \leq \int_{\Omega} \Theta(u_0) dx + \int_{Q_T} G(t, x) |\theta(u_n)|.
\end{aligned}$$

Choosing the constant $\eta \geq \frac{(c(\Lambda))^2}{4d^2}$ in Lemma 4.5, one obtains

$$d \theta'(u_n(t, x)) - c(\Lambda) |\theta(u_n(t, x))| \geq \frac{d}{2} \text{ a.e in } Q_T.$$

On the other hand, $\Theta(u_n(T)) \geq 0$. Therefore,

$$\frac{d}{2} \int_{Q_T} |\nabla u_n|^{p(x)} \leq \int_{\Omega} \Theta(u_0) + \int_{Q_T} G(t, x) |\theta(u_n)|.$$

We may utilize estimate (4.5) to deduce that

$$\int_{Q_T} |\nabla u_n|^{p(x)} \leq C, \quad (4.11)$$

where C is a constant depending only on $\|\underline{u}\|_{\infty}$, $\|\bar{u}\|_{\infty}$ and $\|G\|_{L^1(Q_T)}$. By employing the result of (2.5) in (4.11), we conclude that u_n is uniformly bounded in $\mathcal{V}$. To estimate the nonlinearity (f_n) in $L^1(Q_T)$, we use the growth condition (4.1). We get

$$\begin{aligned}
\int_{Q_T} |f_n(t, x, u_n, \nabla u_n)| & \leq c(|u_n|) \int_{Q_T} \left(G(t, x) + |\nabla u_n|^{p(x)} \right) \\
& \leq c(\Lambda) \int_{Q_T} \left(G(t, x) + |\nabla u_n|^{p(x)} \right).
\end{aligned}$$

Applying the result of (4.11), we conclude that f_n is bounded in $L^1(Q_T)$. Consequently, from the equation satisfied by u_n it follows that $(\partial_t u_n)$ is bounded in $\mathcal{V}^* + L^1(Q_T)$. $\square$

Lemma 4.7. *The sequence (u_n) converges strongly to some u in $\mathcal{V}$.*

Proof. By Lemma 4.6, (u_n) is bounded in $\mathcal{V}$ and $f_n(t, x, u_n, \nabla u_n)$ is bounded in $L^1(Q_T)$. Then, by applying the compactness result of Lemma 5.2, we can extract a subsequence of (u_n) still denoted by (u_n) such that

$$\begin{aligned} (u_n) &\rightarrow u \text{ strongly in } L^{p^-}(Q_T) \text{ and a.e. in } Q_T, \\ (\nabla u_n) &\rightarrow \nabla u \text{ a.e. in } Q_T. \end{aligned}$$

Therefore,

$$A(t, x, \nabla u_n) \rightharpoonup A(t, x, \nabla u) \text{ weakly in } L^{p'(x)}(Q_T).$$

We shall prove that (u_n) converges strongly in $\mathcal{V}$. To do this, we use the difference between the equations satisfied by u_n and u_m . We have

$$\partial_t(u_n - u_m) - \operatorname{div}(A(\nabla u_n)) + \operatorname{div}(A(\nabla u_m)) = f_n(u_n, \nabla u_n) - f_m(u_m, \nabla u_m).$$

Taking $\theta(u_n - u_m) \in \mathcal{V} \cap L^\infty(Q_T)$ as a test function in the weak formulation of the latter equation, one obtains

$$\begin{aligned} &\int_0^T \langle \partial_t(u_n - u_m), \theta(u_n - u_m) \rangle + \int_{Q_T} (A(\nabla u_n) - A(\nabla u_m)) \cdot \nabla(u_n - u_m) \theta'(u_n - u_m) \\ &+ \int_{Q_T} \left(f_n(u_n, \nabla u_n) - f_m(u_m, \nabla u_m) \right) \theta(u_n - u_m) = 0. \end{aligned}$$

Since u_n and u_m have the same initial condition, we have

$$\int_0^T \langle \partial_t(u_n - u_m), \theta(u_n - u_m) \rangle = \int_\Omega \Theta(u_n(T) - u_m(T)) \geq 0.$$

On the other hand, employing the growth condition (4.1), one has

$$\begin{aligned} &\int_{Q_T} (A(\nabla u_n) - A(\nabla u_m)) \cdot \nabla(u_n - u_m) \theta'(u_n - u_m) \\ &\leq c(\Lambda) \int_{Q_T} (G(t, x) + |\nabla u_n|^{p(x)} + |\nabla u_m|^{p(x)}) |\theta(u_n - u_m)|. \end{aligned}$$

Since Θ is positive, by using the coercivity condition (H_2) we get

$$\begin{aligned}
& \int_{Q_T} (A(\nabla u_n) - A(\nabla u_m)) \cdot \nabla(u_n - u_m) \theta'(u_n - u_m) \leq c(\Lambda) \int_{Q_T} G(t, x) |\theta(u_n - u_m)| \\
& \quad + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla u_n |\theta(u_n - u_m)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_m) \cdot \nabla u_m |\theta(u_n - u_m)| \\
& \leq c(\Lambda) \int_{Q_T} G(t, x) |\theta(u_n - u_m)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla(u_n - u_m) |\theta(u_n - u_m)| \\
& \quad + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla u_m |\theta(u_n - u_m)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_m) \cdot \nabla u_n |\theta(u_n - u_m)| \\
& \quad - \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_m) \cdot \nabla(u_n - u_m) |\theta(u_n - u_m)|.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{1}{d} \int_{Q_T} \left(d \theta'(u_n - u_m) - c(\Lambda) |\theta(u_n - u_m)| \right) (A(\nabla u_n) - A(\nabla u_m)) \cdot \nabla(u_n - u_m) \\
& \leq c(\Lambda) \int_{Q_T} G(t, x) |\theta(u_n - u_m)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla u_m |\theta(u_n - u_m)| \\
& \quad + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_m) \cdot \nabla u_n |\theta(u_n - u_m)|.
\end{aligned}$$

Choosing the constant $\eta \geq \frac{c(\Lambda)^2}{4d^2}$ in Lemma 4.5, one has

$$\begin{aligned}
& \frac{1}{2} \int_{Q_T} (A(\nabla u_n) - A(\nabla u_m)) \cdot \nabla(u_n - u_m) \\
& \leq c(\Lambda) \int_{Q_T} G(t, x) |\theta(u_n - u_m)| \\
& \quad + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla u_m |\theta(u_n - u_m)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_m) \cdot \nabla u_n |\theta(u_n - u_m)|.
\end{aligned} \tag{4.12}$$

Due to the fact that $(\nabla u_n) \rightarrow \nabla u$ a.e. in Q_T and $(A(t, x, \nabla u_n)) \rightharpoonup (A(t, x, \nabla u_n))$ weakly in $L^{p'(x)}(Q_T)$, we can use Fatou's Lemma to pass to the limit when m tends to $+\infty$ in (4.12).

We obtain

$$\begin{aligned} \frac{1}{2} \int_{Q_T} (A(\nabla u_n) - A(\nabla u)) \cdot \nabla(u_n - u) &\leq c(\Lambda) \int_{Q_T} G(t, x) |\theta(u_n - u)| \\ &+ \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u_n) \cdot \nabla u |\theta(u_n - u)| + \frac{c(\Lambda)}{d} \int_{Q_T} A(\nabla u) \cdot \nabla u_n |\theta(u_n - u)| \end{aligned}$$

On the other hand, from (H_3) , (4.11), (4.5) and by applying the Lebesgue theorem, we pass to the limit when n tends to $+\infty$ to obtain

$$\lim_{n \rightarrow \infty} \int_{Q_T} (A(\nabla u_n) - A(\nabla u)) \cdot \nabla(u_n - u) \leq 0.$$

Consequently,

$$\nabla u_n \rightarrow \nabla u \text{ strongly in } L^{p(x)}(Q_T).$$

□

4.3. PASSING TO THE LIMIT

In this stage, we will prove that the limit of the sequence (u_n) is a weak solution of the system (4.4) in the sense of Definition 2.3. Thanks to Lemma 4.7, we obtain the existence of a subsequence, still denoted by u_n for simplicity, such that

$$\begin{aligned} \nabla u_n &\rightarrow \nabla u \text{ strongly in } L^{p(x)}(Q_T) \text{ and a.e. in } Q_T, \\ u_n &\rightarrow u \text{ strongly in } L^{p^-}(Q_T) \text{ and a.e. in } Q_T, \\ A(t, x, \nabla u_n) &\rightharpoonup A(t, x, \nabla u) \text{ weakly in } L^{p'(x)}(Q_T), \\ f_n(t, x, u_n, \nabla u_n) &\rightarrow f(t, x, u, \nabla u) \text{ a.e. in } Q_T. \end{aligned}$$

Let us show that

$$f_n(t, x, u_n, \nabla u_n) \rightarrow f(t, x, u, \nabla u) \text{ strongly in } L^1(Q_T)$$

To do this, it suffices to prove that $f_n(t, x, u_n, \nabla u_n)$ is equi-integrable in $L^1(Q_T)$, namely

$$\forall \varepsilon > 0 \exists \delta > 0 \forall E \subset Q_T : |E| < \delta \Rightarrow \int_E |f_n(t, x, u_n, \nabla u_n)| dx dt \leq \varepsilon.$$

Let E be a measurable subset of Q_T and $\varepsilon > 0$, Using the growth assumption (4.1) and (4.5), one has

$$\int_E |f_n(t, x, u_n, \nabla u_n)| \leq \int_E c(\Lambda) \left(G(t, x) + |\nabla u_n|^{p(x)} \right). \quad (4.13)$$

We have $G \in L^1(Q_T)$. Then G is equi-integrable in $L^1(Q_T)$ and therefore there exists $\delta_1 > 0$ such that if $|E| \leq \delta_1$, we have

$$c(\Lambda) \int_E G(t, x) \leq \frac{\varepsilon}{2}.$$

On the other hand, in view of Lemma 4.7, it turn out that $(|\nabla u_n|^{p(x)})$ is equi-integrable in $L^1(Q_T)$, which implies the existence of $\delta_2 > 0$ such that if $|E| \leq \delta_2$, we have

$$c(\Lambda) \int_E |\nabla u_n|^{p(x)} \leq \frac{\varepsilon}{2}$$

By choosing $\delta^* = \inf(\delta_1, \delta_2)$, if $|E| \leq \delta^*$, then it follows that

$$\int_E |f_n(t, x, u_n, \nabla u_n)| \leq \varepsilon.$$

This finishes the proof of Theorem 4.2.

5. APPENDIX

In this section, we propose to prove some auxiliaries results which are useful in the proof of the main result.

Lemma 5.1. *Assume that (H_1) – (H_3) hold, then*

(i) *for any $v_0 \in L^2(\Omega)$ and $g \in L^2(Q_T)$ the following problem*

$$\begin{cases} \partial_t v - \operatorname{div}(A(t, x, \nabla v)) = g(t, x) & \text{in } Q_T, \\ v(0, x) = v_0(x) & \text{in } \Omega, \\ v(t, x) = 0 & \text{on } \Sigma_T \end{cases} \quad (5.1)$$

has a unique solution $v \in \mathcal{V} \cap \mathcal{C}([0, T], L^2(\Omega))$ such that

$$\begin{aligned} \partial_t v &\in \mathcal{V}^* + L^2(Q_T), \quad v(0, x) = v_0(x) \text{ in } L^2(\Omega) \\ \int_0^T \langle \partial_t v, \varphi \rangle + \int_{Q_T} A(t, x, \nabla v) \nabla \varphi &= \int_{Q_T} g(t, x) \varphi, \end{aligned} \quad (5.2)$$

with $\varphi \in \mathcal{V} \cap L^2(Q_T)$.

(ii) *if v is the solution of (5.1), then we have*

$$\|v\|_{\mathcal{V}} + \sup_{0 \leq t \leq T} \|v(t)\|_{L^2(\Omega)} \leq C(\Omega, T) (\|v_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)}), \quad (5.3)$$

$$\|\partial_t v\|_{\mathcal{V}^* + L^2(Q_T)} \leq C(\Omega, T) (\|H\|_{p'(x)} + \|v_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)}). \quad (5.4)$$

Proof. (i) For the existence and uniqueness of the weak solution of the problem (5.1) we refer the reader to [30] and by a direct application of the Aubin–Simon theorem, we deduce that v belongs to $\mathcal{C}([0, T], L^2(\Omega))$ which means that the initial condition makes a sens.

(ii) By choosing $\varphi = v\chi_{(0,t)}$ in (5.2) with $t < T$, one has

$$\frac{1}{2} \int_{\Omega} v^2(t) + \int_{Q_t} A(\tau, x, \nabla v) \nabla v = \frac{1}{2} \int_{\Omega} v_0^2 + \int_{Q_t} v g(\tau, x), \quad (5.5)$$

where $Q_t = (0, t) \times \Omega$. Employing the coercivity assumption (H_2) in (5.5), one has

$$\frac{1}{2} \int_{\Omega} v^2(t) + d \int_{Q_t} |\nabla v|^{p(x)} \leq \frac{1}{2} \int_{\Omega} v_0^2 + \int_{Q_t} v g(\tau, x).$$

As a consequence,

$$\int_{\Omega} v^2(t) \leq \int_{Q_t} g^2(\tau, x) + \int_{Q_t} v^2 + \int_{\Omega} v_0^2. \quad (5.6)$$

By applying Gronwall's lemma, it follows that

$$\int_{Q_T} v^2 \leq (\exp(T) - 1) \left(\|g\|_{L^2(Q_T)}^2 + \int_{\Omega} v_0^2 dx \right).$$

Substituting the above expression in (5.6), one obtains

$$\sup_{0 \leq t \leq T} \int_{\Omega} v^2(t) \leq \|g\|_{L^2(Q_T)}^2 + \exp(T) \left(\|g\|_{L^2(Q_T)} + \int_{\Omega} v_0^2 \right).$$

Then we have

$$\sup_{0 \leq t \leq T} \|v(t)\|_{L^2(\Omega)} \leq C(T, \Omega) \left(\|v_0\|_{L^2(\Omega)} + \|g\|_{L^2(Q_T)} \right). \quad (5.7)$$

By combining (5.5), (5.7) and (H_2) , we deduce that

$$\int_{Q_T} |\nabla v|^{p(x)} dx dt \leq C(T, \Omega) \left(\int_{Q_T} g^2 dx dt + \int_{\Omega} v_0^2 \right). \quad (5.8)$$

By applying the result of (2.5), one gets

$$\|v\|_{\mathcal{V}} \leq C(T, \Omega) (\|g\|_{L^2(Q_T)} + \|v_0\|_{L^2(\Omega)}) \quad (5.9)$$

which implies that v is uniformly bounded in $\mathcal{V}$. Due to the growth assumption (H_1) , we have

$$\begin{aligned} \int_{Q_T} |A(t, x, \nabla v)|^{p'(x)} &\leq C \left(\int_{Q_T} |H(t, x)|^{p'(x)} + \int_{Q_T} |\nabla v|^{p(x)} \right) \\ &\leq C(T, \Omega) \left(\int_{Q_T} |H(t, x)|^{p'(x)} + \int_{Q_T} g^2 + \int_{\Omega} v_0^2 \right). \end{aligned} \quad (5.10)$$

Hence,

$$\|A(t, x, \nabla v)\|_{p'(x)} \leq C(T, \Omega) (\|H\|_{p'(x)} + \|g\|_{L^2(Q_T)} + \|v_0\|_{L^2(\Omega)}). \quad (5.11)$$

To estimate $\partial_t v$ in the norm of the space $\mathcal{V}^* + L^2(Q_T)$, we use the equation satisfied by v . We get

$$\begin{aligned} \|\partial_t v\|_{\mathcal{V}^* + L^2(Q_T)} &\leq C \left(\|A(t, x, \nabla v)\|_{p'(x)} + \|g\|_{L^2(Q_T)} \right) \\ &\leq C(T, \Omega) \left(\|H\|_{p'(x)} + \|g\|_{L^2(Q_T)} + \|v_0\|_{L^2(\Omega)} \right). \quad \square \end{aligned}$$

Lemma 5.2. Assume that (H_1) – (H_4) hold and let $u_n \in \mathcal{V} \cap \mathcal{C}([0, T]; L^2(\Omega))$ be the weak solution of the problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t, x, \nabla u_n)) = f_n(t, x, u_n, \nabla u_n) & \text{in } Q_T, \\ u_n(0, x) = u_0^n(x) & \text{in } \Omega, \\ u_n(t, x) = 0 & \text{on } \Sigma_T \end{cases} \quad (5.12)$$

in the sense that

$$\begin{aligned} \partial_t u_n &\in \mathcal{V}^*, \quad u_n(0, x) = u_0^n(x) \text{ in } L^2(\Omega) \\ \int_0^T \langle \partial_t u_n, \varphi \rangle + \int_{Q_T} A(t, x, \nabla u_n) \nabla \varphi &= \int_{Q_T} f_n(t, x, u_n, \nabla u_n) \varphi \end{aligned} \quad (5.13)$$

for all test function $\varphi \in \mathcal{V}$. If (u_0^n) is bounded in $L^1(\Omega)$, (u_n) is bounded in $\mathcal{V}$ and $(f_n(t, x, u_n, \nabla u_n))$ is bounded in $L^1(Q_T)$. Then, we have (up to a subsequence)

- (i) $u_n \rightarrow u$ strongly in $L^{p^-}(Q_T)$ and a.e. in Q_T ,
- (ii) $\nabla u_n \rightarrow \nabla u$ a.e. in Q_T .

Proof. (i) For s fixed, we have the following embedding relationships:

- (a) if $s > \frac{N}{2}$, we have $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$,
- (b) if $s - 1 > \frac{N}{2}$, one has $H_0^s(\Omega) \hookrightarrow W^{1, p(x)}(\Omega)$, and consequently, $W^{-1, p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$.

On the other hand, (u_n) is bounded in $\mathcal{V}$ and $(f_n(t, x, u_n, \nabla u_n))$ is bounded in $L^1(Q_T)$, and by employing the equation (5.12), it results that $(\partial_t u_n)$ is bounded in $L^1(0, T; H^{-s}(\Omega))$. Furthermore, by using the embedding relationship (2.3), we get that (u_n) is bounded in $L^{p^-}(0, T; W_0^{1, p(x)}(\Omega))$. Moreover, we have

$$W_0^{1, p(x)}(\Omega) \xrightarrow{\text{compact}} L^{p(x)}(\Omega) \hookrightarrow H^{-s}(\Omega).$$

Thanks to the compactness result of Simon (see [31, Corollary 4, p. 85]), we deduce that (up to a subsequence)

$$u_n \rightarrow u \text{ strongly in } L^{p^-}(0, T; L^{p(x)}(\Omega)) \text{ and a.e. in } Q_T.$$

Therefore, the continuous embedding $L^{p^-}(0, T; L^{p(x)}(\Omega)) \hookrightarrow L^{p^-}(Q_T)$ implies that $u_n \rightarrow u$ strongly in $L^{p^-}(Q_T)$ and a.e. in Q_T .

(ii) In this stage, we aim to extend the compactness result of [26] to a more general class of quasilinear parabolic equation with variable exponent. Then to prove the almost everywhere convergence of (∇u_n) we propose to show that (∇u_n) is a Cauchy sequence in measure, namely

$$\forall \delta > 0 \forall \varepsilon > 0 \exists N_0 \forall n, m \geq N_0 : \text{meas} \{(t, x), |(\nabla u_n - \nabla u_m)(t, x)| \geq \delta\} \leq \varepsilon.$$

To do this, let $\delta > 0$ and $\varepsilon > 0$. We remark that for $k > 0$ and $\eta > 0$ the following inequality holds:

$$\begin{aligned} \text{meas} \{(t, x) : |(\nabla u_n - \nabla u_m)(t, x)| \geq \delta\} &\leq \text{meas}(\omega_1) + \text{meas}(\omega_2) \\ &\quad + \text{meas}(\omega_3) + \text{meas}(\omega_4), \end{aligned}$$

where

$$\begin{aligned} \omega_1 &= \{(t, x) : |\nabla u_n| \geq k\}, \\ \omega_2 &= \{(t, x) : |\nabla u_m| \geq k\}, \\ \omega_3 &= \{(t, x) : |u_n - u_m| \geq \eta\} \\ \omega_4 &= \{(t, x) : |(\nabla u_n - \nabla u_m)| \geq \delta, |\nabla u_n| \leq k, |\nabla u_m| \leq k, |u_n - u_m| \leq \eta\}. \end{aligned}$$

To bound $\text{meas}(\omega_1)$ and $\text{meas}(\omega_2)$, we will use the boundedness of u_n and u_m in $\mathcal{V}$. Let us remark that

$$k \text{meas}(\omega_1) \leq \int_{\omega_1} |\nabla u_n| \leq \int_{Q_T} |\nabla u_n|.$$

From assumption (2.1), the following continuous embedding $\mathcal{V} \hookrightarrow L^1(0, T; W_0^{1,1}(\Omega))$ holds true, therefore

$$\text{meas}(\omega_1) \leq \frac{1}{k} \|\nabla u_n\|_{L^1(Q_T)} \leq \frac{C}{k} \|u_n\|_{\mathcal{V}} \leq \frac{C}{k}.$$

By the same manner, one has

$$\text{meas}(\omega_2) \leq \frac{C}{k}.$$

Then, we fix k large enough such that $\text{meas}(\omega_1) \leq \varepsilon$ and $\text{meas}(\omega_2) \leq \varepsilon$. To bound $\text{meas}(\omega_3)$, we will utilize the strong convergence of u_n in $L^{p^-}(Q_T)$. For all $m, n \in \mathbb{N}$, we have

$$\eta \text{meas}(\omega_3) \leq \int_{\omega_3} |u_n - u_m| \leq \int_{Q_T} |u_n - u_m|$$

Using Hölder's inequality, it follows that

$$\text{meas}(\omega_3) \leq \frac{C}{\eta} \|u_n - u_m\|_{L^{p^-}(Q_T)}.$$

On the other hand, from (i) it results that (u_n) is strongly convergent in $L^{p^-}(Q_T)$ which implies that (u_n) is a Cauchy sequence in $L^{p^-}(Q_T)$. Then, for a given η there exists N_0 such that for $m, n \geq N_0$ one gets

$$\text{meas}(\omega_3) \leq \varepsilon.$$

It remains to bound $\text{meas}(\omega_4)$ and to choose η . Due to the assumption (H_3) , one has $[A(t, x, \xi_1) - A(t, x, \xi_2)](\xi_1 - \xi_2) > 0$ for $\xi_1 - \xi_2 \neq 0$. On the other hand, employing the fact that the set

$$\{(\xi_1, \xi_2) \in \mathbb{R}^{2N} : |\xi_1| \leq k, |\xi_2| \leq k \text{ and } |\xi_1 - \xi_2| \geq \delta\}$$

is compact and the function $\xi \mapsto A(t, x, \xi)$ is continuous for almost all (t, x) in Q_T , we deduce that $[A(t, x, \xi_1) - A(t, x, \xi_2)](\xi_1 - \xi_2)$ reaches its minimum on this compact set. Let us denote this minimum by $\gamma(t, x)$. By applying the assumption (H_3) , one has $\gamma(t, x) > 0$ a.e. in Q_T . Moreover, using $\gamma(t, x) > 0$ a.e. in Q_T , we deduce the existence of $\varepsilon' > 0$ such that for all measurable set $\omega \subset Q_T$

$$\int_{\omega} \gamma \leq \varepsilon' \Rightarrow \text{meas}(\omega) \leq \varepsilon. \quad (5.14)$$

Then, to get $\text{meas}(\omega_4) \leq \varepsilon$, it suffices to prove that $\int_{\omega_4} \gamma \leq \varepsilon'$. According to the properties of γ and A , one obtains

$$\int_{\omega_4} \gamma \leq \int_{\omega_4} [A(t, x, \nabla u_n) - A(t, x, \nabla u_m)] (\nabla u_n - \nabla u_m) \mathbf{1}_{\{|u_n - u_m| \leq \eta\}}.$$

It is clearly that

$$\nabla T_{\eta}(u_n - u_m) = (\nabla u_n - \nabla u_m) \mathbf{1}_{\{|u_n - u_m| \leq \eta\}}$$

and thanks to the monotony assumption (H_3) , one has

$$\int_{A_4} \gamma \leq \int_{Q_T} [A(t, x, \nabla u_n) - A(t, x, \nabla u_m)] \nabla T_{\eta}(u_n - u_m).$$

In accordance with (5.13), using the equation satisfied by $(u_n - u_m)$ and choosing

$$\varphi = T_{\eta}(u_n - u_m) \in \mathcal{V} \cap L^{\infty}(Q_T)$$

as a test function, one obtains

$$\begin{aligned} & \int_0^T \langle (u_n - u_m)_t, T_{\eta}(u_n - u_m) \rangle + \int_{Q_T} [A(t, x, \nabla u_n) - A(t, x, \nabla u_m)] \nabla T_{\eta}(u_n - u_m) \\ &= \int_{Q_T} (f_n(t, x, u_n, \nabla u_n) - f_m(t, x, u_m, \nabla u_m)) T_{\eta}(u_n - u_m). \end{aligned}$$

For the first integral, we have

$$\int_0^T \langle (u_n - u_m)_t, T_\eta(u_n - u_m) \rangle = \int_\Omega S_\eta(u_n - u_m)(T) - \int_\Omega S_\eta(u_n - u_m)(0).$$

We remark that $S_\eta(r) \geq 0$ and $S_\eta(r) \leq \eta|r|$, thus

$$\begin{aligned} & \int_{Q_T} [A(t, x, \nabla u_n) - A(t, x, \nabla u_m)] \nabla T_\eta(u_n - u_m) \\ & \leq \eta \int_\Omega |u_0^n - u_0^m| + \eta \int_{Q_T} |f_n(t, x, u_n, \nabla u_n) - f_m(t, x, u_m, \nabla u_m)|. \end{aligned}$$

Since (u_0^n) is bounded in $L^1(\Omega)$ and $(f_n(t, x, u_n, \nabla u_n))$ is bounded in $L^1(Q_T)$, then the last inequality becomes

$$\int_{Q_T} [A(t, x, \nabla u_n) - A(t, x, \nabla u_m)] \nabla T_\eta(u_n - u_m) \leq \eta C.$$

Choosing $\eta \leq \frac{\varepsilon'}{C}$, one obtains $\int_{\omega_4} \gamma \leq \varepsilon'$ and from the result of (5.14), it follows that $\text{meas}(\omega_4) \leq \varepsilon$.

As a consequence, η is fixed and due to boundedness result of $\text{meas}(\omega_3)$, we deduce the existence of $N_0 \in \mathbb{N}$ such that for all $m, n \geq N_0$ we have

$$\text{meas}(\{ |(\nabla u_n - \nabla u_m)(x)| \geq \delta \}) \leq 4\varepsilon.$$

Hence (∇u_n) is a Cauchy sequence in measure. Furthermore, (∇u_n) converges almost everywhere to ∇u in Q_T (up to a subsequence). $\square$

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